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# Measures of noncompactness in Hilbert $C^*$-modules

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**Abstract.** Consider a countably generated Hilbert $C^*$-module $\mathcal{M}$ over a unital $C^*$-algebra $\mathcal{A}$. There is a measure of noncompactness $\lambda$ defined, roughly as the distance from finitely generated sets, which is independent of any topology. We compare $\lambda$ to the Hausdorff measure of noncompactness with respect to the family of seminorms that induce a topology recently introduced by Troitsky, denoted by $\chi$. We obtain $\lambda \equiv \chi$. Related inequalities involving other known measures of noncompactness, e.g. Kuratowski and Istrăţescu are also obtained as well as some related results on adjointable operators.

## 1. Introduction

Hilbert modules, i.e. modules over some $C^*$-algebra have been studied intensively for several decades. Briefly, they are counterparts of Hilbert spaces where the field $\mathbb{C}$ of complex scalars is replaced by some $C^*$-algebra.

*Definition 1.1.* Let $\mathcal{A}$ be some $C^*$-algebra. A pre-Hilbert $C^*$-module is a (right) $C^*$-module $\mathcal{M}$ equipped with a sesquilinear form: $\mathcal{M} \times \mathcal{M} \to \mathcal{A}$ with the following properties:

i) $\langle x, x\rangle \geq 0$ for each $x \in \mathcal{M}$;

ii) $\langle x, x\rangle = 0$ implies that $x = 0$;

iii) $\langle x, y\rangle = \langle y, x\rangle^*$ for each $x, y \in \mathcal{M}$;

iv) $\langle x, ya\rangle = \langle x, y\rangle\, a$ for any $x, y \in \mathcal{M}$ and any $a \in \mathcal{A}$.

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A pre-Hilbert $C^*$-module over $\mathcal{A}$ is a Hilbert $C^*$-module if it is complete with respect to the norm $\|x\| = \| \langle x, x\rangle \|^{\frac{1}{2}}$.

A Cauchy-Schwartz type inequality holds for Hilbert $C^*$-modules ([19, Proposition 1.2.4])

$$\langle x, y\rangle \langle y, x\rangle \leq \|y\|^2 \langle x, x\rangle \quad \text{for each } x, y \in \mathcal{M}. \tag{1.1}$$

*Example 1.2.* Given a unital $C^*$-algebra $\mathcal{A}$, the standard Hilbert module $H_{\mathcal{A}}$ is defined as

$$H_{\mathcal{A}} = \{x = (\xi_1, \xi_2, \ldots) \mid \xi_j \in \mathcal{A}, \sum_{j=1}^{\infty} \xi_j^* \xi_j \text{ converges in the norm topology}\}$$

with the $\mathcal{A}$-valued inner product

$$\langle x, y\rangle = \sum_{j=1}^{\infty} \xi_j^* \eta_j, \quad x = (\xi_1, \xi_2, \ldots), y = (\eta_1, \eta_2, \ldots) \in H_{\mathcal{A}}.$$

Unlike in Hilbert spaces, an $\mathcal{A}$-linear bounded operator on a Hilbert module need not have an adjoint. Therefore, it is usual to consider $B^a(\mathcal{M})$ the set of all bounded, adjointable, $\mathcal{A}$-linear operators on a Hilbert module $\mathcal{M}$.

Among them, $\mathcal{A}$-linear combinations of

$$\Theta_{y,z} : \mathcal{M} \to \mathcal{N}, y \in \mathcal{N}, z \in \mathcal{M}, \qquad \Theta_{y,z}(x) = y \langle z, x\rangle$$

are called finite rank operators. Those belonging to the norm closure of the set of finite rank operators are called "compact" or $\mathcal{A}$-compact. In general, $\mathcal{A}$-compact operators need not map bounded sets into relatively compact sets, as is the case in the framework of Hilbert (and also Banach) spaces, though they share many properties of proper compact operators on a Hilbert space (see [14], [13]). Hence, "compact". Indeed, infinite dimensionality of the image of some bounded operators is contained in the algebra of scalars.

For instance, $P_n : H_{\mathcal{A}} \to H_{\mathcal{A}}$, where $\mathcal{A}$ is unital, given by

$$P_n(\xi_1, \xi_2, \ldots) = (\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots), \tag{1.2}$$

is $\mathcal{A}$-compact. Indeed it can be written as $P_n = \sum_{k=1}^{n} \Theta_{e_k, e_k}$, where $e_k$ is the basic vector consisting of zeros except on $k$-th place it has the unity of $\mathcal{A}$. However, the image of the unit ball in $H_{\mathcal{A}}$ is the unit ball in $\mathcal{A}^n$, which is obviously not compact, unless $\mathcal{A}$ is finite dimensional itself.

For general literature concerning Hilbert modules over $C^*$-algebras, including the standard Hilbert module, the reader is referred to [12], [15] or [24, Part III].

In [9], the authors pose a question of whether there exists a topology on Hilbert module $\mathcal{N}$, where $\mathcal{A}$ is a unital $C^*$-algebra, such that each operator on $M$ is $\mathcal{A}$-compact iff it maps a unit ball (in the norm) to a totally bounded set, and gave a partial answer for the standard Hilbert module, $H_{\mathcal{A}}$ (Example 1.2), by constructing the appropriate topology, here denoted by $\tau_3$.

Soon after, in [23], Troitsky gave a complete answer in the case of countably generated Hilbert $C^*$-modules, constructing a more suitable topology, here denoted by $\tau$.

Both topologies were constructed via a family of seminorms, converting the underlying Hilbert module in a uniform space. This allows us to consider measures of noncompactness that arise from some family of seminorms, in particular Hausdorff, Kuratowski and Istrăţescu measure of noncompactness, see [16] and [1].

Despite any uniform structure, on the standard module $H_{\mathcal{A}}$ there is a natural distance of a given set from finite rank sets, based on [15, Proposition 2.6.2] and the definition preceding it. It was introduced in [10], as follows:

*Definition 1.3.* a) A submodule $L$ of $H_{\mathcal{A}}$ is free finitely generated, if $L$ is isomorphic to $\mathcal{A}^n$, i.e. it is generated by a finite basis.

b) Let $A$, $B \subseteq H_{\mathcal{A}}$. The *excess* of $A$ to $B$, denoted by $\widehat{\mathrm{d}}$, is defined by

$$\widehat{\mathrm{d}}(A,B) := \sup_{a\in A} \operatorname{dist}(a,B) = \sup_{a\in A} \inf_{b\in B} \|a-b\|.$$

c) Let $E \subset H_{\mathcal{A}}$ be a bounded set. The measure of noncompactness of $E$, denoted by $\lambda(E)$, is the greatest lower bound of all $\eta > 0$ for which there exists a free finitely generated module $L \leq H_{\mathcal{A}}$ such that $\widehat{\mathrm{d}}(E,L) < \eta$, i.e.

$$\lambda(E) = \inf_{L} \widehat{\mathrm{d}}(E,L).$$

In [10] there were considered Hausdorff, Kuratowski and Istrăţescu measures of noncompactness, with respect to topology $\tau_3$, or more precisely, the family of seminorms $\mathcal{Q}$ defining it, denoted by $\chi_{\mathcal{Q}}$, $\alpha_{\mathcal{Q}}$ and $I_{\mathcal{Q}}$, respectively, defined by (2.3), (2.4), (2.5) and (2.6). Besides the expected inequalities

$$\chi_{\mathcal{Q}}(E) \leq I_{\mathcal{Q}}(E) \leq \alpha_{\mathcal{Q}}(E) \leq 2\chi_{\mathcal{Q}}(E), \tag{1.3}$$

for any bounded subset $E$ of the standard Hilbert module $H_{\mathcal{A}}$, the following relationships with the measure $\lambda$ were obtained. For an arbitrary bounded $E \subseteq$

$H_{\mathcal{A}}$ there holds

$$\chi_{\mathcal{Q}}(E) \le \lambda(E) \le \sqrt{\|E\| I_{\mathcal{Q}}(E)}, \qquad \|E\| = \sup_{x\in E}\|x\|. \tag{1.4}$$

The first inequality is proved for an arbitrary unital $C^*$-algebra $\mathcal{A}$, whereas the second holds if $\mathcal{A} = B(H)$.

The aim of this note is to examine an arbitrary countably generated Hilbert module $\mathcal{M}$, to give an appropriate definition of $\lambda$ extending those on the standard module $H_{\mathcal{A}}$, and moreover to examine its relationship to the corresponding measures of noncompactness related to the topology $\tau$ defined by Troitsky in [23], i.e. from the family of seminorms $\mathcal{T}$ that define it.

In addition to the expected inequalities (1.3) we prove

$$\lambda(E) = \chi_{\mathcal{T}}(E),$$

for all countably generated Hilbert modules over an arbitrary unital $C^*$-algebra, which is much stronger result than (1.4). Some related results for adjointable operators are also given.

## 2. Prerequisits

In this section we list known results and definitions necessary for the main result.

**2.1. Measure of noncompactness with respect to family of seminorms.** The general theory of noncompactness measures on uniform spaces is developed in [1], [16] and [22]. However, we do not need it in full generality, so we will restrict ourselves to the framework of a Banach space $X$ equipped with a family of seminorms $\{p_i \mid i \in J\}$ which satisfy

$$p_i(x) \le \|x\|, \qquad \text{for all } p_i, x \in X; \tag{2.1}$$

$$p_i(x) = p_i(y), \text{ for all } i \quad \Rightarrow \quad x = y. \tag{2.2}$$

It is well known that these two conditions ensure that the family $\{p_i\}$ determines a locally convex topology weaker than the norm topology. In more details, for any $p_i$, the mapping $X\times X \ni (x,y) \mapsto p_i(x-y) =: d_i(x,y)$ is a pseudometric, i.e. a mapping which differs from metric only in $d_i(x,y) = 0$ need not imply $x = y$.

(However, although a single pseudometric $d_i$ does not separate points, the whole family does, by the second condition.) The family of sets $B_{i,x,\varepsilon}$, $i \in J$, $x \in X$, $\varepsilon > 0$, given by

$$B_{i,x,\varepsilon} = B_i(x;\varepsilon) = \{y \in X \mid p_i(x-y) < \varepsilon\}$$

forms the subbasis for the corresponding topology.

Uniform structure is usually defined via so called *entourages*, see [3, p. 169]. In our case, the entourages can be defined by seminorms as

$$E_{i,\varepsilon} := \{(x,y) \in X \times X \mid p_i(x-y) < \varepsilon\}.$$

However, the uniform spaces can be defined directly by a family of pseudometrics, see [4, Theorem 1, p.142].

*Definition 2.1.* A set $E \subset X$ is *totally bounded* if for every $p_i$ and every $\varepsilon > 0$ there exists a finite collection $y_1, y_2, \ldots, y_n$ of elements of $E$ such that the sets $B_i(y_j;\varepsilon) = \{x \in E \mid p_i(x-y_j) < \varepsilon\}$, $j = 1, 2, \ldots, n$ form a cover of $E$.

*Remark 2.2.* In the usual framework, where a uniform space is defined by entourages, there is also another different definition of totally bounded sets. These two definitions are equivalent, see [11, page 198].

Three most common measures of noncompactness on metric spaces are Hausdorff measure $\chi$, Kuratowski measure $\alpha$ and Istrăţescu measure $I$. Nothing is lost if we replace metric by some pseudometric, e.g. $(x,y) \mapsto p_i(x-y)$. Hence, for a given seminorm $p_i$ we have

$$\chi_{p_i}(E) = \inf\{\varepsilon > 0 \mid E \subseteq \bigcup_{j=1}^{m} B_i(x_j,\varepsilon),\ \text{for some } x_j \in X\}, \tag{2.3}$$

$$\alpha_{p_i}(E) = \inf\{\varepsilon > 0 \mid E = \bigcup_{j=1}^{m} E_j,\ \operatorname{diam}_{p_i}(E_j) := \sup_{x,y\in E_j} p_i(x-y) < \varepsilon\}, \tag{2.4}$$

$$I_{p_i}(E) = \sup\{\varepsilon > 0 \mid \exists \text{ infinite } S \subset E \text{ so that } \forall\, x,y \in S, x \neq y \Rightarrow p_i(x-y) \geq \varepsilon\}. \tag{2.5}$$

The following properties of $\chi$, $\alpha$ and $I$ are well known. Their proof can be found in [20] and [2].

**Proposition 2.3.** *Let $E$, $E_1$, $E_2$ be bounded subsets of some Banach space $X$, and let $\mu$ denote either $\chi$ or $\alpha$ or $I$. Then*

*(1) if $E_1 \subseteq E_2$ then $\mu(E_1) \le \mu(E_2)$;*

*(2) $\mu(E_1 + E_2) \le \mu(E_1) + \mu(E_2)$;*

*(3) $\chi(E) \le I(E) \le \alpha(E) \le 2\chi(E)$.*

*Definition 2.4.* Let $X$ be a Banach space and let $\{p_i \,|\, i \in J\}$ be a family of seminorms that satisfy (2.1) and (2.2). Also, let $\chi_i$, $\alpha_i$, $I_i$ be the measures of noncompactness corresponding to the seminorm $p_i$ (or more precisely to the pseudometric $d_i(x,y) = p_i(x-y)$). We define $\chi$, $\alpha$ and $I$ as

$$\chi(E) = \sup_{i\in J} \chi_i(E), \quad \alpha(E) = \sup_{i\in J} \alpha_i(E), \quad I(E) = \sup_{i\in J} I_i(E). \tag{2.6}$$

If we want to emphasize the family of seminorms, e.g. $\mathcal{P}$, we shall write $\chi_{\mathcal{P}}$, $\alpha_{\mathcal{P}}$ or $I_{\mathcal{P}}$.

The following properties of such defined measure $\mu$ are easy to derive. Moreover, they were proved in more generality in [1, 16].

**Proposition 2.5.** *Let $X$ be a Banach space with a family of seminorms $\{p_i \,|\, i \in J\}$ and let $\mu$ be either of measures defined by (2.6). Then $\mu$ has the following properties:*

*(1) $\mu(E) = +\infty$ if and only if $E$ is unbounded for at least one $p_i$;*

*(2) $\mu(E) = \mu(\overline{E})$;*

*(3) $\mu(E) = 0$ if and only if $E$ is totally bounded set;*

*(4) from $E \subseteq F$ it follows $\mu(E) \le \mu(F)$;*

*(5) $\mu(E \cup \{x\}) = \mu(E)$.*

*Remark 2.6.* Although only one direction of item (3) is proved in [1] and [16], the other is valid for $\chi$, $\alpha$ and $I$.

**Lemma 2.7.** *For any family of seminorms $\mathcal{P}$ there holds*

$$\chi_{\mathcal{P}}(E) \le I_{\mathcal{P}}(E) \le \alpha_{\mathcal{P}}(E) \le 2\chi_{\mathcal{P}}(E).$$

Proof. It follows immediately from Proposition 2.3 (3) and the definition. □

Note that measures of noncompactness defined by (2.6) highly depends on the family of seminorms and can differ even if two families determine the same topology. However, in a specific but rather general enough framework we can speak about measures of noncompactness with respect to locally convex topology.

*Definition 2.8.* Let $X$ be a Banach space and let $\tau$ be some locally convex topology on $X$ weaker than the norm topology. Consider the set of all seminorms on $X$ contionuous with respect to $\tau$ and less or equal to the norm. Such a set of seminorms we shall call *the canonical set of seminorms.*

It is easy to see that the canonical set of seminorms generates the initial topology. For details, see [21, a corollary after 1.14, 1.38(b)]. Such a set of seminorms is defined naturally and only using the topology. Hence, it is canonical.

*Definition 2.9.* Let $X$ be a Banach space, and let $\tau$ be some locally convex topology weaker than the norm topology. We say that the measures of noncompactness $\chi$, $\alpha$, and $I$ defined by (2.6) with respect to the canonical set of seminorms are measures of noncompactness with respect to the topology $\tau$. We shall denote them by $\chi_\tau$, $\alpha_\tau$, and $I_\tau$, respectively.

**2.2. Measure of noncompactness $\lambda$.** The measure of noncompactness $\lambda$ defined by Definition 1.3 on the standard Hilbert module $H_{\mathcal{A}}$ was studied in [10]. Among others, the following properties of $\lambda$ were proved.

**Proposition 2.10.** *The measure of noncompactness $\lambda$ has the following properties*

*(1)* $\lambda(E) = \inf_{M\in\mathcal{F}} \sup_{x\in E} d(x, M) = \lim_{n\to\infty} \sup_{x\in E} \|x - P_n x\|$, *where $\mathcal{F}$ is the set of all free finitely generated modules and $P_n \colon H_{\mathcal{A}} \to H_{\mathcal{A}}$ is given by* $P_n(x_1, x_2, \dots) = (x_1, x_2, \dots, x_n, 0, 0, \dots)$;

*(2)* *if $E \subset F$, then* $\lambda(E) \le \lambda(F)$;

*(3)* $\lambda(E + F) \le \lambda(E) + \lambda(F)$;

*(4)* $\lambda(B_1) = 1$, *where $B_1$ is the unit ball in $H_{\mathcal{A}}$;*

*(5)* $\lambda(E) \le \sup_{x\in E} \|x\|$;

*(6)* $\lambda(E) = 0$ *if and only if $E$ is $\mathcal{A}$-precompact.*

**2.3. Troitsky's topology.**

*Definition 2.11.* Let $\mathcal{N}$ be a Hilbert $C^*$-module over $\mathcal{A}$. A countable system $X = \{x_i\}$ of its elements is called admissible for a (possibly non-closed) submodule $\mathcal{N}^0 \subset \mathcal{N}$ (or $\mathcal{N}^0$-admissible) if

1) for every $x \in \mathcal{N}^0$ the series $\sum_i \langle x, x_i\rangle \langle x_i, x\rangle$ is convergent;

2) the sum in the previous item is bounded by $\langle x, x\rangle$;

3) $\|x_i\| \le 1$ for each $i$.

**Lemma 2.12.** *Let $\mathcal{N}$, $\mathcal{N}^0$ be as in the previous definition, and let $\Phi$ be a countable collection $\{\varphi_1, \varphi_2, \dots\}$ of states on $\mathcal{A}$. Further, let $X = \{x_i\}$ be an $\mathcal{N}^0$-admissible system. The function $p_{X,\Phi}$ defined by*

$$p_{X,\Phi}(x) = \sqrt{\sup_k \sum_{i=k}^{\infty} |\varphi_k(\langle x, x_i\rangle)|^2}, \quad x \in \mathcal{N}^0 \tag{2.7}$$

*is well-defined seminorm on $\mathcal{N}^0$ and $p_{X,\Phi}(x) \leq \|x\|$.*

Proof. It was shown in [23, Lemma 2.5, Theorem 2.6]. □

*Definition 2.13.* Let $\mathcal{N}$ be a Hilbert $C^*-$module over a unital $C^*$-algebra $\mathcal{A}$, and let $\mathcal{N}^0$ be some (possibly non closed) submodule of $\mathcal{N}$. Troitsky's topology $\tau$ (or more informatively $(\mathcal{N}, \mathcal{N}^0)$-topology) on the submodule $\mathcal{N}^0$ is the locally convex topology defined by the family of seminorms (2.7), where $X = \{x_i\}$ is $\mathcal{N}^0$-admissible and $\Phi = \{\varphi_1, \varphi_2, \dots\}$ is a countable collection of states on $\mathcal{A}$.

The family of seminorms (2.7) that defines Troitsky's topology we shall call *Troitsky's family* and we shall denote it by $\mathcal{T}$.

By $\bar{\mathcal{T}}$ denote the family of all seminorms on $\mathcal{N}^0$ which are continuous with respect to the topology induced by $\mathcal{T}$ and which are bounded above by the norm.

The family of seminorms $\mathcal{T}$, which determine the uniform structure on $\mathcal{N}^0$ was initially introduced in [23].

The family $\bar{\mathcal{T}}$ is the canonical family of the Troitsky's topology in the sense of Definition 2.8, and hence $\mathcal{T}$ and $\bar{\mathcal{T}}$ generate the same topology. Moreover, we have

**Proposition 2.14.** *The family $\bar{\mathcal{T}}$ consists of exactly those seminorms $p$ on $\mathcal{N}^0$ that satisfy:*

1) *$p(x) \leq \|x\|$ for all $x \in \mathcal{N}^0$;*
2) *there are $q_1$, …, $q_n \in \mathcal{T}$ and $C > 0$ such that $p \leq C \max\{q_1, \dots, q_n\}$.*

*Consequently $\mathcal{T} \subseteq \bar{\mathcal{T}}$. Also $\mathcal{T}$ and $\bar{\mathcal{T}}$ generate the same uniformity.*

Proof. Indeed, if $p \in \bar{\mathcal{T}}$ then $\{x \in \mathcal{N}^0 \mid p(x) < 1\}$ is a neighborhood of zero and hence it contains a basic neighborhood of zero (for the topology generated by $\mathcal{T}$). In other words there are $q_i \in \mathcal{T}$, $1 \leq i \leq n$ and $c_i > 0$ such that

$$\bigcap_{i=1}^{n} \{x \in \mathcal{N}^0 \mid q_i(x) < c_i\} \subseteq \{x \in \mathcal{N}^0 \mid p(x) < 1\},$$

or equivalently

$$\max_i \frac{1}{c_i} q_i(x) < 1 \Rightarrow p(x) < 1.$$

From the latter, using positive homogeneity of $p$, $q_i$, one can easily derive the second condition. The first is obvious. Conversely, if some seminorm fulfills 1) and 2) then it is, also obviously, continuous with respect to $\mathcal{T}$.

The last conclusion follows from 1) and 2). □

The uniform structure induced by $\mathcal{T}$ and also by $\bar{\mathcal{T}}$ perfectly fits the notion of $\mathcal{A}$-compact operators.

**Theorem 2.15.** [23, Theorem 2.13] *Suppose, $F : \mathcal{M} \to \mathcal{N}$ is an adjointable morphism of Hilbert $C^*$-modules over $C^*$-algebra $\mathcal{A}$. Then $F$ is $\mathcal{A}$-compact if and only if $F(B)$ is $(\mathcal{N},\mathcal{N})$-totally bounded with respect to uniform structure $\tau$, where $B$ is the unit ball of $M$.*

Although [23] didn't deal with sets, but only with operators, the following statement is easy to derive.

**Corollary 2.16.** *Let $H_{\mathcal{A}}$ be the standard Hilbert $C^*$-module over a unital $C^*$-algebra $\mathcal{A}$ and let $P_n : H_{\mathcal{A}} \to H_{\mathcal{A}}$ be the projection given by (1.2). If $E$ is a norm bounded set, then $P_n(E)$ is totally bounded with respect to $\tau$.*

Proof. We have $P_n(x) = \sum_{k=1}^n e_k \langle e_k, x\rangle$, for a standard basis $\{e_n\}$ of $H_{\mathcal{A}}$, hence $P_n$ is $\mathcal{A}$-compact. If $E$ is norm bounded then $E \subseteq tB$, for a suitable $t > 0$. Hence $P_n(E) \subseteq tP_n(B)$ which is totally bounded. □

Finally, we quote a simple and well known property of positive elements.

**Lemma 2.17.** [17, Lemma 3.3.6] *If $a$ is a normal element of a non-zero $C^*$-algebra $\mathcal{A}$, then there is a state $\varphi$ of $\mathcal{A}$ such that $|\varphi(a)| = \|a\|$.*

## 3. Main results

### 3.1. Standard Hilbert module.

**Lemma 3.1.** *Let $H_{\mathcal{A}}$ be the standard Hilbert module over $\mathcal{A}$, and let $\mathcal{P} = \{p_i \mid i \in J\}$ be a family of seminorms such that $p_i(x) \le \|x\|$ for every $i \in J$, $x \in H_{\mathcal{A}}$, which turns $H_{\mathcal{A}}$ into a locally convex space.*

*Let $B_1$ denote the unit ball in $H_{\mathcal{A}}$. Then $\chi_{\mathcal{P}}(B_1) \le 1$. In particular, $\chi_{\mathcal{T}}(B_1)$, $\chi_{\bar{\mathcal{T}}}(B_1) \le 1$.*

Proof. Indeed, from $p_i(x) \le \|x\|$ we have $B_1 \subseteq \{x \in H_{\mathcal{A}} \mid p_i(x) < 1+\varepsilon\}$ for any $i$ and any $\varepsilon > 0$. $\square$

**Lemma 3.2.** *Let $H_{\mathcal{A}}$ be the standard Hilbert module over $\mathcal{A}$, and let $\mathcal{P} = \{p_i \mid i \in J\}$ be a family of seminorms which satisfies (2.1) and (2.2).*

*Let $P_n : H_{\mathcal{A}} \to H_{\mathcal{A}}$ denote the projection (1.2) and let $E \subset H_{\mathcal{A}}$ be a norm bounded set. If $P_nE$ is totally bounded (with respect to all $p_i$) for every $n \in \mathbb{N}$, then*

$$\mu_{\mathcal{P}}(E) \le \lambda(E)\mu_{\mathcal{P}}(B_1), \qquad B_1 \text{ unit ball in } H_{\mathcal{A}}, \tag{3.1}$$

*for every measure of noncompactness $\mu \in \{\chi_{\mathcal{P}}, \alpha_{\mathcal{P}}, I_{\mathcal{P}}\}$ defined by (2.6).*

*In particular, (3.1) holds for the family of seminorms $\mathcal{T}$ defined by (2.7). i.e. for $\mu \in \{\chi_{\mathcal{T}}, \alpha_{\mathcal{T}}, I_{\mathcal{T}}\}$.*

*Concerning Hausdorff measure of noncompactness, by virtue of Lemma 3.1, we have*

$$\chi_{\mathcal{P}}(E) \le \lambda(E). \tag{3.2}$$

Proof. Let E be a bounded set. Since $E \subset P_nE + (I-P_n)E$, and since $\mu$ is subadditive and monotone (Proposition 2.3), we have

$$\mu(E) \le \mu(P_nE) + \mu((I-P_n)E).$$

Since, by assumption $\mu(P_nE) = 0$ we have

$$\mu(E) \le \mu((I-P_n)E) \le \sup_{x \in E} \|(I-P_n)x\|\mu(B_1),$$

for all $n \in \mathbb{N}$. (The last inequality follows from $(I-P_n)(E) \subseteq \sup_{x\in E}\|(I-P_n)x\|B_1$.) Therefore, by Proposition 2.10 (1), $\mu(E) \le \lambda(E)\mu(B_1)$.

By Corollary 2.16 $P_nE$ is totally bounded with respect to seminorms (2.7), whereas by Proposition 2.3, Theorem 2.5 and Remark 2.6 all $\chi_{\mathcal{P}}$, $\alpha_{\mathcal{P}}$ and $I_{\mathcal{P}}$ are subadditive, monotone and satisfy $\mu(F) = 0$ if and only if $F$ is totally bounded. $\square$

**Lemma 3.3.** *Let $H_{\mathcal{A}}$ be the standard Hilbert $C^*$-module over $\mathcal{A}$ and $E \subset H_{\mathcal{A}}$. Let $Y = \{y_i\}, y_i \in E$ be a countable system and $(n_k)_{k\ge1}$ be an increasing sequence in $\mathbb{N}$. Then*

$$X = \{z_i\}, \quad z_i = \|P_{n_i}(I-P_{n_{i-1}})y_i\|^{-1}P_{n_i}(I-P_{n_{i-1}})y_i, \quad i \ge 1$$

*is $H_{\mathcal{A}}$-admissible, where $P_n$ is defined by (1.2), and $n_0 = 0$, $P_0 = 0$.*

Proof. 1) Let $x = (x_1, x_2, \ldots) \in H_{\mathcal{A}}$ and $\varepsilon > 0$ be arbitrary. Since $\sum_{i=1}^{\infty} x_i^* x_i$ is convergent in norm there exists $m_0 \in \mathbb{N}$ such that for any $m > m_0$ and $p_1 \in \mathbb{N}$ we have

$$\left\| \sum_{i=m}^{m+p_1} x_i^* x_i \right\| < \varepsilon.$$

Denote $T_i = P_{n_i}(I - P_{n_{i-1}})$; then $z_i = \|T_i y_i\|^{-1} T_i y_i$ and hence, taking into account that $T_i$ are selfadjoint projections:

$$\begin{aligned} \langle x, z_i \rangle \langle z_i, x \rangle &= \left\langle x, \|T_i y_i\|^{-1} T_i y_i \right\rangle \left\langle \|T_i y_i\|^{-1} T_i y_i, x \right\rangle \\ &= \left\langle T_i x, \|T_i y_i\|^{-1} T_i y_i \right\rangle \left\langle \|T_i y_i\|^{-1} T_i y_i, T_i x \right\rangle. \end{aligned}$$

However, by Cauchy Schwartz inequality (1.1),

$$\left\langle T_i x, \|T_i y_i\|^{-1} T_i y_i \right\rangle \left\langle \|T_i y_i\|^{-1} T_i y_i, T_i x \right\rangle \leq \langle T_i x, T_i x \rangle.$$

Hence

$$\begin{aligned} \left\| \sum_{i=m}^{m+p} \langle x, z_i \rangle \langle z_i, x \rangle \right\| &\leq \left\| \sum_{i=m}^{m+p} \langle T_i x, T_i x \rangle \right\| = \\ &= \left\| \sum_{i=n_{m-1}+1}^{n_{m+p}} x_i^* x_i \right\| = \left\| \sum_{i=n_{m-1}+1}^{n_{m-1}+p_1} x_i^* x_i \right\|, \end{aligned}$$

for $p_1 = n_{m+p} - n_{m-1} - 1$. Since $n_{m-1} + 1 \geq m$, the last term does not exceed $\varepsilon$. Therefore, $\sum_{i=1}^{\infty} \langle x, z_i \rangle \langle z_i, x \rangle$ is convergent in the norm.

2) For every $x \in H_{\mathcal{A}}$

$$\sum_i \langle x, z_i \rangle \langle z_i, x \rangle \leq \sum_i \langle T_i x, T_i x \rangle = \langle x, x \rangle.$$

The last equality holds, since $T_i x = (0, \ldots, 0, x_{n_{i-1}+1}, \ldots, x_{n_i}, 0, \ldots)$.

3) Obviously, $\|z_i\| \leq 1$ for each $i$. □

**Theorem 3.4.** *Let $E \subset H_{\mathcal{A}}$ be a bounded set, let $u \in A$ be a unitary and let $\mu$ stand for any of Hausdorff, Kuratowski or Istrăţescu measure of noncompactness. Then $\mu_{\mathcal{T}}(Eu) = \mu_{\mathcal{T}}(E)$. (Here, $\mathcal{T}$ denotes the Troitsky's family of seminorms (2.7).)*

Proof. Let $\varepsilon > 0$ be arbitrary. There exists an $H_{\mathcal{A}}$-admissible pair $(X, \Phi)$, $X = \{x_i\}$, $\Phi = \{\varphi_i\}$ such that $\mu_{p_{X,\Phi}}(Eu) > \mu(Eu) - \varepsilon$. Let $\varphi_i^u(x) = \varphi_i(u^* x u)$

and $x_i^u = x_i u^*$ for $i \in \mathbb{N}$. Obviously, $\varphi_i^u(1) = 1$ and $\varphi_i^u(x) \geq 0$ whenever $x \geq 0$, so $\varphi_i^u$ are states on $H_{\mathcal{A}}$. Since

$$\langle x, x_i^u \rangle \langle x_i^u, x \rangle = \langle x, x_i \rangle u^* u \langle x_i, x \rangle = \langle x, x_i \rangle \langle x_i, x \rangle,$$

and

$$\|x_i^u\| = \|x_i u^*\| \leq \|x_i\| \cdot \|u^*\| = \|x_i\|,$$

the pair $(X^u, \Phi^u)$ is $H_{\mathcal{A}}$-admissible, where $\Phi^u = \{\varphi_1^u, \varphi_2^u, \ldots\}$ and $X^u = \{x_1^u, x_2^u, \ldots\}$.

Thus

$$\begin{aligned} p_{X,\Phi}^2(xu) &= \sup_k \sum_{i=k}^{+\infty} |\varphi_k(\langle xu, x_i \rangle)|^2 = \sup_k \sum_{i=k}^{+\infty} |\varphi_k(u^* \langle x, x_i \rangle)|^2 \\ &= \sup_k \sum_{i=k}^{+\infty} |\varphi_k(u^* \langle x, x_i u^* \rangle u)|^2 = \sup_k \sum_{i=k}^{+\infty} |\varphi_k^u(\langle x, x_i u^* \rangle)|^2 \\ &= \sup_k \sum_{i=k}^{+\infty} |\varphi_k^u(\langle x, x_i^u \rangle)|^2 = p_{X^u,\Phi^u}^2(x). \end{aligned}$$

Therefore, $\mu_{p_{X^u,\Phi^u}}(E) > \mu(Eu) - \varepsilon$ and hence $\mu_{\mathcal{T}}(E) \geq \mu_{\mathcal{T}}(Eu)$. The opposite inequality follows by $E = (Eu)u^{-1}$. □

**Theorem 3.5.** *Let $\mathcal{T}$ and $\bar{\mathcal{T}}$ be families of seminorms defined in Definition 2.13. For any bounded set $E \subset H_{\mathcal{A}}$, we have*

$$\chi_{\mathcal{T}}(E) = \chi_{\bar{\mathcal{T}}}(E) = \lambda(E).$$

Proof. By Lemma 3.2 we have $\chi_{\bar{\mathcal{T}}}(E) \leq \lambda(E)$. Since $\mathcal{T} \subseteq \bar{\mathcal{T}}$ we have $\chi_{\mathcal{T}}(E) \leq \chi_{\bar{\mathcal{T}}}(E)$. Therefore, it is enough to prove $\lambda(E) \leq \chi_{\mathcal{T}}(E)$.

Let $\lambda = \lambda(E) = \inf_{i \geq 1} \sup_{x \in E} \|(I - P_i)x\|$ and let $0 < \varepsilon < \lambda$ be arbitrary. Suppose that the Hausdorff measure of noncompactness $\chi(E)$ is at most $\lambda - \varepsilon$, that is for every pair $(X, \Phi)$ with $H_{\mathcal{A}}$-admissible $X$ there are $y_1, y_2, \ldots, y_l \in H_{\mathcal{A}}$ such that for every $y \in E$ there exists $k \in \{1, 2, \ldots, l\}$ such that $p_{X,\Phi}(y - y_k) < \lambda - \varepsilon$.

Since $\sup_{x \in E} \|(I - P_i)x\| = \lambda$, there is a sequence $z_i \in E$ such that $\|(I - P_i)z_i\| > \lambda - \varepsilon/4$. Further, there is an increasing sequence $i_j$ of positive integers for which $\|P_{i_{j+1}} z_{i_j} - z_{i_j}\| < \varepsilon/4$. We have

$$\begin{aligned} \|(P_{i_{j+1}} - P_{i_j}) z_{i_j}\| &= \|(P_{i_j} z_{i_j} - z_{i_j}) - (P_{i_{j+1}} z_{i_j} - z_{i_j})\| \\ &> \|(I - P_{i_j}) z_{i_j}\| - \|P_{i_{j+1}} z_{i_j} - z_{i_j}\| > \lambda - \varepsilon/4 - \varepsilon/4 \\ &= \lambda - \varepsilon/2. \end{aligned}$$

Choose $x_j = \|(P_{i_{j+1}} - P_{i_j})z_{i_j}\|^{-1}(P_{i_{j+1}} - P_{i_j})z_{i_j}$. By Lemma 3.3 $X_0 = \{x_1, x_2, \dots\}$ is $H_{\mathcal{A}}$-admissible.

Note that

$$\left\langle x_j, (P_{i_{j+1}} - P_{i_j})z_{i_j} \right\rangle = \|(P_{i_{j+1}} - P_{i_j})z_{i_j}\| \langle x_j, x_j\rangle \geq 0,$$

as well as

$$\| \left\langle x_j, (P_{i_{j+1}} - P_{i_j})z_{i_j} \right\rangle \| = \|(P_{i_{j+1}} - P_{i_j})z_{i_j}\| > \lambda - \varepsilon/2.$$

Therefore, by Lemma 2.17 there are states $\varphi_j$ such that

$$|\varphi_j(\left\langle x_j, (P_{i_{j+1}} - P_{i_j})z_{i_j} \right\rangle)| > \lambda - \varepsilon/2.$$

For the seminorm $p_{X_0,\Phi_0} \in \mathcal{T}$, $\Phi_0 = \{\varphi_1, \varphi_2, \ldots\}$, let $y_1, y_2, \ldots, y_l \in H_{\mathcal{A}}$ be the corresponding $\lambda - \varepsilon$ net. One can find a number $j_0$ such that

$$\|(I - P_{i_j})y_k\| < \varepsilon/4 \quad j \geq j_0, \quad k = 1, \ldots, l$$

and hence, for $j \geq j_0$, $k = 1, 2, \ldots, l$, using Cauchy-Schwarz inequality (1.1)

$$\begin{aligned} \| \langle y_k, x_j\rangle \| &= \| \left\langle \|(P_{i_{j+1}} - P_{i_j})z_{i_j}\|^{-1}(P_{i_{j+1}} - P_{i_j})z_{i_j}, (P_{i_{j+1}} - P_{i_j})y_k \right\rangle \| \\ &\leq \|(P_{i_{j+1}} - P_{i_j})y_k\| < \varepsilon/2. \end{aligned}$$

Then, for all $k = 1, \ldots, l$ and $y = z_{i_j}$,

$$\begin{aligned} p_{X_0,\Phi_0}(y - y_k) &= \sqrt{\sup_m \sum_{i=m}^{\infty} |\varphi_m(\langle y - y_k, x_i\rangle)|^2} \\ &\geq |\varphi_j(\left\langle z_{i_j} - y_k, x_j \right\rangle)| \geq |\varphi_j(\left\langle z_{i_j}, x_j \right\rangle)| - |\varphi_j(\langle y_k, x_j\rangle)| \\ &= |\varphi_j(\left\langle (P_{i_{j+1}} - P_{i_j})z_{i_j}, x_j \right\rangle)| - |\varphi_j(\langle y_k, x_j\rangle)| > \lambda - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} \\ &= \lambda - \varepsilon. \end{aligned}$$

This contradicts the choice of $y_1, \ldots, y_l$. Hence, the assumption that there is a $\lambda - \varepsilon$ net can not hold. Thus, $\lambda(E) \leq \chi_{\mathcal{T}}(E)$. □

*Remark 3.6.* Following [10, Theorem 4.7], one might prove the inequality $\lambda(E) \leq \sqrt{\|E\| I_{\mathcal{T}}(E)}$. However, due to the previous theorem, it is trivial. Indeed, $\lambda(E) = \chi_{\mathcal{T}}(E) \leq \|E\|, I_{\mathcal{T}}(E)$.

**Corollary 3.7.** *Let $E$ be a bounded set in the standard Hilbert module $H_{\mathcal{A}}$. There holds*

$$\chi_{\mathcal{T}}(E) = \lambda(E) \le I_{\mathcal{T}}(E) \le \alpha_{\mathcal{T}}(E) \le 2\chi_{\mathcal{T}}(E) = 2\lambda(E),$$

*and also*

$$\chi_{\bar{\mathcal{T}}}(E) = \lambda(E) \le I_{\bar{\mathcal{T}}}(E) \le \alpha_{\bar{\mathcal{T}}}(E) \le 2\chi_{\bar{\mathcal{T}}}(E) = 2\lambda(E),$$

Proof. Follows immediately from Lemma 2.7 and Theorem 3.5. □

**Corollary 3.8.** *Let $B_1$ be the closed unit ball of $H_{\mathcal{A}}$. Then $\chi_{\mathcal{T}}(B_1) = \chi_{\bar{\mathcal{T}}}(B_1) = 1$.*

Proof. Follows immediately from Proposition 2.10 and Theorem 3.5. □

**3.2. Arbitrary countably generated Hilbert module.** We want to extend our results from the standard Hilbert module $H_{\mathcal{A}}$ to an arbitrary countably generated Hilbert module $\mathcal{M}$ over a unital $C^*$-algebra $\mathcal{A}$. As it is usual, we use the Kasparov stabilization theorem $\mathcal{M} \oplus H_{\mathcal{A}} \cong H_{\mathcal{A}}$.

Note that the measure of noncompactness $\lambda$ is defined via free finitely generated submodules which was suitable for the standard Hilbert module. In more general purpose the lack of free submodules forces us to find another definition of $\lambda$.

Let $\mathcal{M}$ be a Hilbert module over a $C^*$-algebra $\mathcal{A}$, and let $x_1, x_2, \dots, x_n \in \mathcal{M}$. Their $\mathcal{A}$-linear span $\mathrm{Lin}_{\mathcal{A}}(x_1, \dots, x_n) = \{x_1a_1 + \dots + x_na_n \mid a_j \in \mathcal{A}\}$, need not to be closed. For a Hilbert submodule $\mathcal{N} \subseteq \mathcal{M}$ we say that it is *algebraically* finitely generated if it coincide with $\mathrm{Lin}_{\mathcal{A}}(x_1, \dots, x_n)$ for some $x_1, \dots, x_n$, and we say that it is *topologicaly* finitely generated if some $\mathrm{Lin}_{\mathcal{A}}(x_1, \dots, x_n)$ is dense in $\mathcal{N}$. Similarly for countably generated.

*Remark 3.9.* Occasionally, some authors drop words "algebraically" and "topolgicaly" to simplify the text. However, it should be noted that *finitely generated* by rule means *algebraically finitely generated*, whereas *countably generated* by rule means *topologically countably generated.*

Among (algebraically) finitely generated submodules, the most regular are *projective* ones. We say that $L \subseteq \mathcal{M}$ is *finitely generated projective submodule* of $\mathcal{M}$ if it is isomorphic to another module $L_1$ which is a direct summand in a finitely generated free module, i.e. $L \cong L_1$, $L_1 \oplus L_2 = \mathcal{A}^n$. In [24, Proposition 15.4.2] several equivalent conditions to be projective finitely generated module are described.

The following Lemma is very well known, but we stated and proved it in the lack of an appropriate reference.

**Lemma 3.10.** *Let $L$ be an algebraically finitely generated and projective submodule of some countably generated Hilbert module $\mathcal{M}$ over a unital algebra $\mathcal{A}$. Then:*

*a) There is a selfadjoint projection $P : \mathcal{M} \to \mathcal{M}$ such that $P(\mathcal{M}) = L$;*

*b) If, moreover, $\mathcal{M} = H_{\mathcal{A}}$, then $P$ is a compact operator.*

Proof. a) By [15, Theorem 1.4.6], $\mathcal{M} = L \oplus L^{\perp}$, i.e. $L$ is orthogonally complementable. Therefore, there is a projection $P : H_{\mathcal{A}} \to H_{\mathcal{A}}$ such that $P(\mathcal{M}) = L$. Indeed, for $\mathcal{M} \ni x = y + z$, where $y \in L$, $z \in L^{\perp}$ are unique, take $Px = y$. Then, it is easy to verify that $P$ is linear and norm decreasing. Further, if $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$, $y_1$, $y_2 \in L$, $z_1$, $z_2 \in L^{\perp}$ then both $\langle Px_1, x_2\rangle$ and $\langle x_1, Px_2\rangle$ are equal to $\langle y_1, y_2\rangle$. Hence $P^* = P$ since adjoint if exists it is unique [24, Lemma 15.2.3].

b) For $\mathcal{M} = H_{\mathcal{A}}$, the compactness of $P$ is actually proved in the proof of $3 \Rightarrow 4$ from [24, Theorem 15.4.2]. Without repeating details, the idea is this: Consider the compact operator $Kx = \sum_{i=1}^{m} y_i \langle x, e_i\rangle$, where $y_j$, $1 \le j \le m$ are generators for $L$, and $e_i$ make the canonical basis for $H_{\mathcal{A}}$. Then $K$ has a closed range $L$ and has the polar decomposition $K = V|K|$. Then $P = VV^*$ is compact.

□

*Remark 3.11.* Part b) of the previous Lemma can be proved for any countably generated $\mathcal{M}$ using Kasparov stabilization. However we don't need it.

**Lemma 3.12.** *Let $E$ be a bounded subset of $H_{\mathcal{A}}$.*

*a) We have*

$$\lambda(E) = \lim_{n\to\infty} \sup_{x\in E} \|x - P_n x\| = \inf_{M\in\mathcal{P}} \widehat{\mathrm{d}}(E, M) = \inf_{M\in\mathcal{P}} \sup_{x\in E} d(x, M),$$

*where $P_n$ are defined by (1.2), and $\mathcal{P}$ denotes the set of all finitely generated projective submodules of $H_{\mathcal{A}}$;*

*b) For $x_1, \ldots, x_n \in H_{\mathcal{A}}$ and $M > 0$, let $L_{x_1,\ldots,x_n,M} = \{x_1 a_1 + \cdots + x_n a_n \mid a_j \in \mathcal{A}, \|a_j\| \le M\}$. Denote $\mathcal{L} = \{L_{x_1,\ldots,x_n,M} \mid x_j \in H_{\mathcal{A}}, M > 0\}$. Then*

$$\lambda(E) = \inf_{L\in\mathcal{L}} \widehat{\mathrm{d}}(E, L) = \inf_{L\in\mathcal{L}} \sup_{x\in E} d(x, L).$$

Proof. a) The first equality is stated in Proposition 2.10 (1) (proved in [10, Proposition 2.2]) and literally the same argument leads to the second. Since it is short we repeat it.

Denote

$$\widehat{\lambda}(E) = \inf_{M\in\mathcal{P}} \widehat{\mathrm{d}}(E, M).$$

Since $P_nH_\mathcal{A}$ is finitely generated and projective (and moreover free), we have $\widehat{\lambda}(E) \le \sup_{x\in E}\|x - P_nx\|$ and hence $\widehat{\lambda}(E) \le \lim_{n\to\infty}\sup_{x\in E}\|x - P_nx\|$.

For the opposite inequality, let $\varepsilon > 0$ be arbitrary. Then, there is a finitely generated projective module $L \subseteq H_\mathcal{A}$ such that $\sup_{x\in E} d(x, L) < \widehat{\lambda}(E) + \varepsilon$. By Lemma 3.10, the corresponding projection $P_L$ is $\mathcal{A}$-compact, and by [15, Proposition 2.2.1] $\|P_L(I - P_n)\| \to 0$, as $n \to \infty$. Then

$$\begin{aligned}\|x - P_nx\| = d(x, P_nH_\mathcal{A}) \le \|x - P_nP_Lx\| \le\\ \le \|x - P_Lx\| + \|P_Lx - P_nP_Lx\| \le \widehat{\lambda}(E) + \varepsilon + \|E\|\,\|P_L - P_nP_L\|,\end{aligned}$$

and taking $\sup_{x\in E}$ and limit we obtain the desired inequality.

b) Denote

$$\lambda_1(E) = \inf_{L\in\mathcal{L}} \widehat{\mathrm{d}}(E, L).$$

Let $\varepsilon > 0$ be arbitrary. Then, there are $x_j \in H_\mathcal{A}$, $M > 0$ such that for $L = L_{x_1,\dots,x_m,M} \in \mathcal{L}$ we have

$$\sup_{x\in E} d(x, L) < \lambda_1(E) + \varepsilon/4. \tag{3.3}$$

Further, for all $x \in E$ there are $a_j^{(x)} \in \mathcal{A}$ such that $\|a_j^{(x)}\| \le M$ and

$$\|x - (x_1a_1^{(x)} + \dots + x_ma_m^{(x)})\| \le d(x, L) + \varepsilon/4. \tag{3.4}$$

Let $n \in \mathbf{N}$ and $x \in E$ be arbitrary. We have

$$\begin{aligned}\|x - P_nx\| =& d(x, P_n(H_\mathcal{A})) \le d(x, P_n(L)) =\\ =& \inf_{\substack{a_1,\dots,a_m\in\mathcal{A}\\ \|a_j\|\le M}} \|x - \sum_{j=1}^m P_n(x_j)a_j\| \le \|x - \sum_{j=1}^m P_n(x_j)a_j^{(x)}\| \le\\ \le& \|x - \sum_{j=1}^m x_ja_j^{(x)}\| + \|\sum_{j=1}^m (x_j - P_n(x_j))a_j^{(x)}\|.\end{aligned}$$

The first summand is bounded by $\lambda_1(E) + \varepsilon/2$ by (3.3) and (3.4). For the second we have:

$$\|\sum_{j=1}^m (x_j - P_n(x_j))a_j^{(x)}\| \le M\sum_{j=1}^m \|x_j - P_nx_j\| < \frac{\varepsilon}{2},$$

for all $n$ large enough (i.e. $n \geq n_0$ for some $n_0$), independently of $x$. Thus, taking the supremum over all $x \in E$ we obtain $\sup_{x\in E} \|x - P_n x\| \leq \lambda_1(E) + \varepsilon$ for $n$ large enough. Finally, passing to the limit we obtain $\lambda(E) \leq \lambda_1(E) + \varepsilon$ and consequently $\lambda(E) \leq \lambda_1(E)$ since $\varepsilon$ was arbitrary.

Conversely, let $\varepsilon$ be arbitrary and let $n \in \mathbf{N}$ be such that $\sup_{x\in E} \|x - P_n x\| < \lambda(E) + \varepsilon$. For such $n$ consider the set

$$L_n = \{e_1 a_1 + \cdots + e_n a_n \mid a_i \in \mathcal{A}, \|a_i\| \leq 2\|E\|\},$$

where $e_k$ is the basic vector for $H_{\mathcal{A}}$, i.e. its $k$th entry is the unity of $\mathcal{A}$ and other entries are zero.

We claim that for all $x \in E$, $P_n x \in L_n$. Indeed, $P_n x$ can be written as $e_1 x_1 + \cdots + e_n x_n$, where $x = (x_1, x_2, \dots)$. Then, we have

$$\|x_j\| = \|(P_j - P_{j-1})x\| \leq \|P_j x\| + \|P_{j-1} x\| \leq 2\|x\| \leq 2\|E\|.$$

Since $L_n \subseteq P_N(H_{\mathcal{A}})$,, we have $\|x - P_n x\| = d(x, P_n(H_{\mathcal{A}})) \geq d(x, L_n)$. If we apply a supremum over all $x \in E$, we will get

$$\lambda(E) + \varepsilon > \sup_{x\in E} \|x - P_n x\| \geq \sup_{x\in E} d(x, L_n) \geq \lambda_1(E),$$

because $L_n \in \mathcal{L}$. □

The bounded scalars in the definition of the family $\mathcal{L}$ might seem artificial and superfluous. A more natural idea would be to consider merely the family of finitely generated submodules (regardless of whether they are closed or not) and to define

$$\lambda_2(E) = \inf \widehat{\mathrm{d}}(E, L), \tag{3.5}$$

where the infimum is taken over all finitely generated submodules $L$.

However, the following example shows that it is not right way, i.e. what can happen when scalars are not bounded, even if the observed set is bounded.

*Example 3.13.* Let $\mathcal{A}$ be a unital $C^*$-algebra, and let $\varphi_j \in \mathcal{A}$ be norm-one elements such that $\varphi_i \varphi_j = \varphi_j \varphi_i = 0$ for all $i \neq j$. (Such a family exists in various algebras. For instance, in $B(H)$ with $H$ infinite dimensional, we can take a family of mutually orthogonal rank one projections. In $C(K)$, we can take test functions whose supports are mutually disjoint.) Consider the vector $x = \sum \frac{1}{2^j} e_j \varphi_j = (\varphi_1/2, \varphi_2/4, \dots, \varphi_n/2^n, \dots)$, and the "scalars" $a_j = 2^j \varphi_j$.

Then the set $E = \{x a_j \mid j \in \mathbf{N}\}$ is contained in a finitely generated submodule $\mathrm{Lin}(x)$ and hence $\lambda_2(E) = 0$, where $\lambda_2$ is defined by (3.5). On the other hand, $xa_j = e_j\varphi_j = (0, \dots, 0, \varphi_j, 0, \dots)$ and therefore $xa_j - P_n xa_j = xa_j = e_j\varphi_j$ for $n < j$ and $xa_j - P_n xa_j = 0$ for $n \geq j$. This implies $\sup_{y\in E} \|y - P_n y\| = 1$ for all $n$, or $\lambda(E) = 1$.

Note that $E$ is bounded even though the scalars are not.

Both quantities $\widehat{\lambda}$ and $\lambda_1$ from Lemma 3.12 can be used to define the measure of noncompactness on an arbitrary countably generated module. The second one $\lambda_1$ occurs to be better, since it behaves well with respect to orthogonal sums.

*Definition 3.14.* Let $\mathcal{M}$ be a countably generated Hilbert module over a unital $C^*$-algebra $\mathcal{A}$, and let $E \subseteq \mathcal{M}$ be a bounded set. We define

$$\lambda(E) = \inf_{L\in\mathcal{L}} \sup_{x\in E} d(x, L),$$

where $\mathcal{L}$ is defined by

$$\mathcal{L} = \{L_{x_1,x_2,\dots,x_m,M} \mid x_j \in \mathcal{M}, M > 0\},$$

$$L_{x_1,x_2,\dots,x_m,M} = \{\sum_{j=1}^{m} x_j a_j \mid a_j \in \mathcal{A}, \|a_j\| \leq M\}.$$

*Remark 3.15.* Of course, the previous definition can be extended to modules that are not countably generated. However, in this case, we can not use Kasparov stabilization theorem in the proof of Theorem 3.23.

**Proposition 3.16.** *Let $\mathcal{M}_1 \subseteq \mathcal{M}_2$ be two countably generated Hilbert modules over the same $C^*$-algebra $\mathcal{A}$, such that $\mathcal{M}_1$ is complemented in $\mathcal{M}_2$, i.e.*

$$\mathcal{M}_2 = \mathcal{M}_1 \oplus \mathcal{N} \tag{3.6}$$

*for some $\mathcal{N} \subseteq \mathcal{M}_2$ and let $E \subseteq \mathcal{M}_1$ be a bounded set.*

*a) Consider two functions $\lambda_j(E)$, $j = 1, 2$, the measures of noncompactness regarding $E$ as a subset of $\mathcal{M}_j$. Then $\lambda_1(E) = \lambda_2(E)$;*

*b) The same is valid for $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ if $\mathcal{M}_1$ is selfdual and $\mathcal{A}$ is a $W^*$-algebra.*

Proof. a) Obviously, $\lambda_1(E) \geq \lambda_2(E)$ since in the latter the infimum is taken over a larger set. Conversely, let $\varepsilon > 0$ be arbitrary. Then, there is some $L \subseteq \mathcal{M}_2$ of the form

$$L = L_{x_1,\dots,x_m,M} = \{x_1a_1 + \dots + x_m a_m \mid a_j \in \mathcal{A}, \|a_j\| \leq M\}, \qquad x_j \in \mathcal{M}_2$$

such that $\sup_{x\in E} d(x, L) < \lambda_2(E) + \varepsilon$.

By (3.6) there exists the projection $Q : \mathcal{M}_2 \to \mathcal{M}_1$. Then

$$Q(L_{x_1,\dots,x_m,M}) = L_{Qx_1,\dots,Qx_m,M}, \qquad Qx_j \in \mathcal{M}_1.$$

Now, let $x \in E \subseteq \mathcal{M}_1$ and $y \in L$. Then $x = Qx$ and therefore

$$\begin{aligned}\langle x-y, x-y\rangle &= \langle Q(x-y)-(I-Q)y, Q(x-y)-(I-Q)y\rangle = \\ &= \langle Q(x-y), Q(x-y)\rangle + \langle (I-Q)y, (I-Q)y\rangle \geq \langle Q(x-y), Q(x-y)\rangle .\end{aligned}$$

Hence

$$\|x-y\| \geq \|Qx-Qy\| = \|x-Qy\| \geq d(x, Q(L)).$$

Taking the infimum over all $y \in L$ we obtain $d(x,L) \geq d(x,Q(L))$, and taking the supremum over $x \in E$ we get

$$\sup_{x\in E} d(x,L) \geq \sup_{x\in E} d(x,Q(L)) \geq \lambda_1(E),$$

which finally leads to $\lambda_2(E) \geq \lambda_1(E)$.

b) This part can be proved in exactly the same way as the previous one. The only difference is that it is not obvious that $Q(L)$ is (algebraically) finitely generated projective submodule of $\mathcal{M}_1$ whenever $L$ is of that kind.

Let $L \subseteq \mathcal{M}_2$ is finitely generated and projective and let $x_j$, $1 \leq j \leq m$ be its generators. Then $Q(L) = \mathrm{Lin}_{\mathcal{A}}(Qx_1,\dots,Qx_m)$. Denote $y_j = Qx_j$. Since $\mathcal{M}_1$ is selfdual and $\mathcal{A}$ is a $W^*$-algebra, by [15, Proposition 3.4.1] there are vectors $z_j \in \mathcal{M}_1$ such that $y_j = z_j \langle y_j, y_j\rangle^{1/2}$ and $p_j = \langle z_j, z_j\rangle$ is a projection onto the range of $\langle y_j, y_j\rangle^{1/2}$. (In fact, the minimal projection $p$ for which $p\langle y_j, y_j\rangle^{1/2} = \langle y_j, y_j\rangle^{1/2}$.)

Then $\sum_{j=1}^m y_j a_j = \sum_{j=1}^m z_j \langle y_j, y_j\rangle^{1/2} a_j = \sum_{j=1}^m z_j b_j$ for some $b_j \in \mathcal{A}$. Hence

$$Q(L) = \mathrm{Lin}_{\mathcal{A}}(y_1,\dots,y_m) \subseteq \mathrm{Lin}_{\mathcal{A}}(z_1,\dots,z_m) = \mathrm{Lin}_{\mathcal{A}}(z_1)+\dots+\mathrm{Lin}_{\mathcal{A}}(z_m). \quad (3.7)$$

Now, let us prove that $\mathrm{Lin}_{\mathcal{A}}(z_j)$ is a projective module. For $z_j a$, $z_j b \in \mathrm{Lin}_{\mathcal{A}}(z_j)$ we have

$$\langle z_j a, z_j b\rangle = a^* \langle z_j, z_j\rangle b = a^* p_j b = (p_j a)^* p_j b.$$

Let us recall that every right ideal in $\mathcal{A}$ is a Hilbert module over $\mathcal{A}$ with inner product $(a,b) \mapsto a^* b$, [15, Example 1.3.3]. Hence, the mapping $\mathrm{Lin}_{\mathcal{A}}(z_j) \ni z_j a \mapsto$

$p_j a \in p_j\mathcal{A}$ preserves the inner product and is thus an isometry. It is obviously $\mathcal{A}$-linear and bijective. Therefore, $\mathrm{Lin}_{\mathcal{A}}(z_j) \cong p_j\mathcal{A}$. However, $\mathcal{A} = p_j\mathcal{A} \oplus (1-p_j)\mathcal{A}$, from which we conclude that $\mathrm{Lin}_{\mathcal{A}}(z_j)$ is isomorphic to a direct summand in a free finitely generated module (actually 1 generated), i.e. that it is projective.

Since $\mathcal{A}$ is a $W^*$-algebra, by [15, Lemma 3.6.10] a sum of two projective finitely generated submodules is also projective and finitely generated. Therefore $\mathrm{Lin}_{\mathcal{A}}(z_1) + \dots + \mathrm{Lin}_{\mathcal{A}}(z_m)$ is finitely generated projective. By (3.7), $Q(L)$ is contained in some finitely generated projective submodule, which is enough to finish the proof. □

The previous Proposition establishes that the measure $\lambda$ does not depend on the ambient module, i.e. that it is an intrinsic function of the set. Next, we want to prove that similar independence result holds for Hausdorff measure $\chi$. We start with an abstract Proposition.

**Proposition 3.17.** *Let $Z$ be a linear space, and let $Z_1$, $Z_2 \leq Z$ be its linear subspaces such that $Z$ is their direct sum. Let $q$ be a seminorm on $Z$ and let $q_i$ be a seminorm on $Z_i$, $i = 1, 2$. Let $p_i : Z \to Z_i$ be the canonical linear projection. For the Hausdorff measure of noncompactness we have*

*a) If for all $z \in Z$ we have $q(z) \leq q_1(p_1 z) + q_2(p_2 z)$ then for all bounded $E \subseteq Z$ we have $\chi_q(E) \leq \chi_{q_1}(p_1 E) + \chi_{q_2}(p_2 E)$;*

*b) If for all $z \in Z$ we have $q(z) \geq \max\{q_1(p_1 z), q_2(p_2 z)\}$ then $\chi_q(E) \geq \max\{\chi_{q_1}(p_1 E), \chi_{q_2}(p_2 E)\}$.*

Proof. a) Let $\delta_1 > \chi_{q_1}(p_1 E)$ and $\delta_2 > \chi_{q_2}(p_2 E)$ be arbitrary. Then there are finite $\delta_1$-net for $p_1 E$ in seminorm $q_1$, i.e. there are $u_1, u_2, \dots, u_m \in Z_1$ such that

$$\bigcup_{i=1}^{m} \{u \in Z_1 \mid q_1(u - u_i) < \delta_1\} \supseteq p_1 E.$$

Similarly, there are $v_1, v_2, \dots, v_s \in Z_2$ such that

$$\bigcup_{j=1}^{s} \{v \in Z_2 \mid q_2(v - v_j) < \delta_2\} \supseteq p_2 E.$$

We claim that $u_i + v_j$ make a finite $\delta_1 + \delta_2$ net for $E$ in the seminorm $q$. Indeed, if $y \in E$, then $y = p_1 y + p_2 y$, where $p_i y \in p_i E$. Let $i$ and $j$ be such that $q_1(p_1 y - u_i) < \delta_1$ and $q_2(p_2 y - v_j) < \delta_2$. Then

$$q(y - (u_i + v_j)) = q((p_1 y - u_i) + (p_2 y - v_j)) \leq q_1(p_1 y - u_i) + q_2(p_2 y - v_j) < \delta_1 + \delta_2.$$

Hence $\chi_q(E) \leq \delta_1 + \delta_2$.

b) Let $\delta > \chi(E)$ be arbitrary. Then there are $w_1, w_2, \ldots, w_n \in Z$ such that

$$\bigcup_{i=1}^{n} \{z \in Z \mid q(z - w_i) < \delta\} \supseteq E.$$

We claim that $p_1 w_i$ make a finite $\delta$-net for $p_1 E$ in the seminorm $q$. Indeed an arbitrary $y \in p_1$ is of the form $y = p_1 z$ for some $z \in E$, and for an appropriate $i$ we have

$$q_1(y - p_1 w_i) = q_1(p_1(z - w_i)) \leq q(z - w_i) < \delta.$$

Hence $\chi_{q_1}(p_1 E) < \delta$. Similarly for $\chi_{q_2}$. □

We shall apply the previous Proposition twice. First, we can derive inequalities for $\chi_{\mathcal{T}}$ where $\mathcal{T}$ is Troitsky's family od seminorms.

**Corollary 3.18.** *Let $\mathcal{M}_j$, $j = 1, 2$ be two countably generated modules and let $\mathcal{M}_j^0$ be their submodules. Also, let $\chi_{\mathcal{T}}$ be Hausdorff measure of noncompactness with respect to Troitsky's family $\mathcal{T}$ of seminorms of the form (2.7), where $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ and $\mathcal{M}^0 = \mathcal{M}_1^0 \oplus \mathcal{M}_2^0$. Also, let $\chi_{\mathcal{T}}^j$ be the corresponding Hausdorff measures of noncompactness related to $(\mathcal{M}_j; \mathcal{M}_j^0)$.*

*Denote by $p_j$ projection from $\mathcal{M}$ onto $\mathcal{M}_j$. If $E \subseteq \mathcal{M}$ is a bounded set, then*

$$\max\{\chi_{\mathcal{T}}^1(p_1 E), \chi_{\mathcal{T}}^2(p_2 E)\} \leq \chi_{\mathcal{T}}(E) \leq \chi_{\mathcal{T}}^1(p_1 E) + \chi_{\mathcal{T}}^2(p_2 E). \tag{3.8}$$

*Moreover, $p_j$ is $(\mathcal{M}_j; \mathcal{M}_j^0) - (\mathcal{M}; \mathcal{M}^0)$ continuous.*

Proof. Denote by $J_j = p_j^* : \mathcal{M}_j \to \mathcal{M}$ the corresponding inclusions.

Let $X = \{x_i\}$ be an arbitrary sequence in $\mathcal{M}$ admissible for $\mathcal{M}^0$ and let $\Phi = \{\varphi_i\}$ be an arbitrary sequence of states on $\mathcal{A}$. It is easy to see that $X^{(j)} = \{p_j x_i\}$ is admissible for $\mathcal{M}_j^0$, for details see the proof of [23, Lemma 2.15], as well as (for $j = 1, 2$)

$$p_{X,\Phi}(J_j y) = \sqrt{\sup_{k \geq 1} \sum_{i=k}^{\infty} |\varphi_k(\langle J_j y, x_i \rangle)|^2} = \sqrt{\sup_{k \geq 1} \sum_{i=k}^{\infty} |\varphi_k(\langle y, p_j x_i \rangle)|^2} = p_{X^{(j)},\Phi}(y).$$

Thus, the triple of norms $q = p_{X,\Phi}$, $q_1 = p_{X^{(1)},\Phi}$ and $q_2 = p_{X^{(2)},\Phi}$ fulfills the assumptions of the first part of the Proposition 3.17, and hence

$$\chi_{p_{X,\Phi}}(E) \leq \chi_{p_{X^{(1)},\Phi}}(p_1 E) + \chi_{p_{X^{(2)},\Phi}}(p_2 E) \leq \chi_{\mathcal{T}}^1(p_1 E) + \chi_{\mathcal{T}}^2(p_2 E),$$

for an arbitrary bounded $E$, or $\chi_{\mathcal{T}}(E) \leq \chi_{\mathcal{T}}^1(p_1 E) + \chi_{\mathcal{T}}^2(p_2 E)$, after passing to the supremum over all $p_{X,\Phi}$.

This proves the second inequality in (3.8).

Let $\Phi = \{\varphi_i\}$ be an arbitrary sequence of states on $\mathcal{A}$ and let $X = \{x_i\}$ be an arbitrary sequence in $\mathcal{M}_1$ admissible for $\mathcal{M}_1^0$. Then it is easy to see that $J_1X = \{J_1x_i\}$ is admissible for $\mathcal{M}^0$ as well as

$$p_{X,\Phi}(p_1y) = \sqrt{\sup_{k\geq 1}\sum_{i=k}^{\infty}|\varphi_k(\langle p_1y, x_i\rangle)|^2} = \sqrt{\sup_{k\geq 1}\sum_{i=k}^{\infty}|\varphi_k(\langle y, J_1x_i\rangle)|^2} = p_{J_1X,\Phi}(y). \tag{3.9}$$

The triple of norms $q = p_{J_1X,\Phi}$, $q_1 = p_{X,\Phi}$, $q_2 = 0$ satisfy the assumptions of the second part of Proposition 3.17 and hence $\chi_{p_{X,\Phi}}(p_1E) \leq \chi_{p_{J_1X,\Phi}}(E) \leq \chi_{\mathcal{T}}(E)$, for all bounded $E$, or $\chi^1_{\mathcal{T}}(p_1E) \leq \chi_{\mathcal{T}}(E)$ after passing to the supremum over all $X$ and $\Phi$.

Moreover, from (3.9) we can easily derive that $p_1$ is $(\mathcal{M}, \mathcal{M}^0) - (\mathcal{M}_1, \mathcal{M}_1^0)$ continuous.

Interchanging roles of $\mathcal{M}_1$ and $\mathcal{M}_2$ we can obtain a similar result for $\chi^2_{\mathcal{T}}$, and the first inequality in (3.8) is proved. □

The previous Corollary, more precisely the inequalities (3.8), will be enough to establish that $\chi_{\mathcal{T}}$ does not depend on ambient space. We shall prove it in Corollary 3.22.

Meantime, we apply Proposition 3.17 to abstract results that will lead to the independence of $\chi_{\tilde{\mathcal{T}}}$, i.e. to Hausdorff measure of noncompactness with respect to Troitsky's topology. (To topology itself, and not merely w.r.t. some family of seminorms.)

**Proposition 3.19.** *Let $Z$ be a normed space, and let $Z_1$, $Z_2 \leq Z$ be its linear subspaces such that $Z$ is their direct sum. Let $\tau$ be some topology on $Z$ weaker that the norm topology. Suppose also, that the canonical projections $p_i : Z \to Z_i$, $i = 1, 2$ are of norm 1, and that are $\tau$-$\tau|_{Z_i}$ continuous.*

*It is easy to see that the restrictions $\tau_1$ and $\tau_2$ of $\tau$ to $Z_1$ and $Z_2$ respectively are also weaker than the norm topology. Hence all of them have canonical families of seminorms. Let $\chi_\tau$, $\chi_{\tau_1}$ and $\chi_{\tau_2}$ denote the corresponding Hausdorff measures of noncompactness (see Definition 2.9).*

*Then*

$$\max\{\chi_{\tau_1}(p_1E), \chi_{\tau_2}(p_2E)\} \leq \chi_\tau(E) \leq \chi_{\tau_1}(p_1E) + \chi_{\tau_2}(p_2E). \tag{3.10}$$

Proof. Let us recall that $\chi_\tau = \chi_{\mathcal{P}}$ where $\mathcal{P}$ is the canonical family of seminorms with respect to $\tau$ - see Definition 2.8.

Let $q$ be a seminorm continuous with respect to $\tau$ and bounded above by the norm. Then the restriction $q_1 = q|_{Z_1}$ is bounded above by the norm and continuous with respect to $\tau_1$. Analogously for $q_2$. Since obviously, for $z = z_1 + z_2$, $z_i \in Z_i$ we have

$$q(z) = q(z_1 + z_2) \leq q(z_1) + q(z_2) = q_1(z_1) + q_2(z_2),$$

we have, by the first part of Proposition 3.17 that

$$\chi_q(E) \leq \chi_{q_1}(p_1E) + \chi_{q_2}(p_2E) \leq \chi_{\tau_1}(p_1E) + \chi_{\tau_2}(p_2E),$$

and consequently the second inequality in (3.10).

For the first inequality, pick a seminorm $q_1$ on $Z_1$ bounded above by the norm and continuous with respect to $\tau_1$. Let $q = q_1 \circ p_1$. Then, $q$ is continuous with respect to $\tau$, since $p_1$ $\tau$-$\tau_1$ continuous. Also, $q$ is bounded above by the norm, since $\|p_1\| = 1$.

Therefore $q \geq \max\{q_1, 0\}$ and by the first part of Proposition 3.17 we get

$$\chi_{q_1}(p_1E) \leq \chi_q(E) \leq \chi_\tau(E),$$

and consequently $\chi_{\tau_1}(p_1E) \leq \chi_\tau(E)$ by taking a supremum over all $q_1$ from the canonical family of seminorms.

Similar can be done for $\chi_{\tau_2}$ which yields the first inequality in (3.10). □

Before we derive the inequality (3.10) for $\mathcal{T}$, we need a simple technical lemma concerning topology.

**Lemma 3.20.** *Let $(X_1, \tau_1)$, $(X_2, \tau_2)$ and $(X, \tau)$ be three topological spaces such that $X = X_1 \times X_2$ as a set. Consider the set $X_1 \times X_2$ as a topological space with the product topology $\tau_1 \times \tau_2$.*

*a) If identity $\mathrm{id} : (X_1 \times X_2, \tau_1 \times \tau_2) \to (X, \tau)$ and projections $p_j : (X, \tau) \to (X_j, \tau_j)$, $j = 1, 2$ are continuous then $\tau = \tau_1 \times \tau_2$.*

*b) If, moreover, $X_1$ and $X_2$ are locally convex spaces, then $\tau_1 \times \{0\} = \tau|_{X_1 \times \{0\}}$ and $\{0\} \times \tau_2 = \tau|_{\{0\} \times X_2}$.*

Proof. a) Indeed, the product topology is the weakest topology such that the corresponding projections are continuous. Hence, $\tau \supseteq \tau_1 \times \tau_2$. On the other hand, by continuity of identic mapping we have $\tau \subseteq \tau_1 \times \tau_2$.

b) Let $G \times \{0\} \in \tau_1 \times \{0\}$. Since $p_1$ is continuous, we have $G \times X_2 \in \tau$. From $G \times \{0\} = (X_1 \times \{0\}) \cap (G \times X_2)$ we get $G \times \{0\} \in \tau|_{X_1 \times \{0\}}$.

Let $G \times \{0\} \in \tau|_{X_1 \times \{0\}}$. Then $G \times \{0\} = (X_1 \times \{0\}) \cap W$ for some $W \in \tau$. However, since $\tau = \tau_1 \times \tau_2$, $W = \bigcup_{j \in J}(G_j \times V_j)$, for some $G_j \in \tau_1$ and $V_j \in \tau_2$. Then

$$G \times \{0\} = (X_1 \times \{0\}) \cap W = (X_1 \times \{0\}) \cap \bigcup_{j \in J}(G_j \times V_j) = \bigcup_{j \in J} G_j \times \{0\} \in \tau_1 \times \{0\}.$$

Hence $\tau_1 \times \{0\} = \tau|_{X_1 \times \{0\}}$. Similarly for $\{0\} \times \tau_2 = \tau|_{\{0\} \times X_2}$. □

**Corollary 3.21.** *Let $\mathcal{M}_j$, $j = 1, 2$ be two countably generated modules and let $\mathcal{M}_j^0$ be their submodules. Also, let $\chi_{\tilde{\mathcal{T}}}$ be Hausdorff measure of noncompactness of the family $\tilde{\mathcal{T}}$ defined in Definition 2.13, where $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ and $\mathcal{M}^0 = \mathcal{M}_1^0 \oplus \mathcal{M}_2^0$. Also, let $\chi_{\tilde{\mathcal{T}}}^j$ be the corresponding Hausdorff measures of noncompactness related to $(\mathcal{M}_j; \mathcal{M}_j^0)$ topology.*

*Denote by $p_j$ projection from $\mathcal{M}$ onto $\mathcal{M}_j$. If $E \subseteq \mathcal{M}$ is a bounded set, then*

$$\max\{\chi_{\tilde{\mathcal{T}}}^1(p_1 E), \chi_{\tilde{\mathcal{T}}}^2(p_2 E)\} \leq \chi_{\tilde{\mathcal{T}}}(E) \leq \chi_{\tilde{\mathcal{T}}}^1(p_1 E) + \chi_{\tilde{\mathcal{T}}}^2(p_2 E). \tag{3.11}$$

Proof. By Corollary 3.18, projections $p_j$, as well as the identity mapping $\mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}$ are continuous. Hence, by Lemma 3.20, $(\mathcal{M}_1, \mathcal{M}_1^0)$ Troitsky's topology is the restriction of Troitsky's $(\mathcal{M}, \mathcal{M}^0)$ topology.

Now, it is enough to apply Proposition 3.19. □

Finally, we are in a position to prove the independence of both $\chi_{\mathcal{T}}$ and $\chi_{\tilde{\mathcal{T}}}$ on ambient module.

**Corollary 3.22.** *Let $\mathcal{M}_1 \subseteq \mathcal{M}_2$ be two countably generated Hilbert modules over $\mathcal{A}$, such that $\mathcal{M}_1$ is complemented in $\mathcal{M}_2$, and let $E \subseteq \mathcal{M}_1$ be an arbitrary bounded set.*

*Then*

$$\chi_{\mathcal{T}}^1(E) = \chi_{\mathcal{T}}^2(E),$$
$$\chi_{\tilde{\mathcal{T}}}^1(E) = \chi_{\tilde{\mathcal{T}}}^2(E),$$

*where $\chi_{\mathcal{T}}^j$ denote the Hausdorff measure of noncompactness for Troitsky's family of seminorm with respect to $\mathcal{M}_j$, and similar for $\chi_{\tilde{\mathcal{T}}}^1$.*

Proof. After some changes in notations we can apply the previous proposition with $\mathcal{M}_j^0 = \mathcal{M}_j$ and $E \subseteq \mathcal{M}_1$. Then $p_2 E$ is trivial, implying $\chi_{\mathcal{T}}^2(p_2 E) = 0$. Thus the equation (3.8) becomes

$$\chi_{\mathcal{T}}^1(E) \leq \chi_{\mathcal{T}}^2(E) \leq \chi_{\mathcal{T}}^1(E) + 0.$$

The second equality we obtain by using (3.11) instead of (3.8). □

**Theorem 3.23.** *Let $\mathcal{M}$ be a countably generated Hilbert $C^*$-module over a unital $C^*$-algebra $\mathcal{A}$. Let $\chi_{\mathcal{T}}$ denote the Hausdorff measure of noncompactness with respect to Troitsky's family of seminorms $\mathcal{T}$, and let $\lambda$ denote the measure of noncompactness defined in Definition 3.14.*

*If $E \subseteq \mathcal{M}$ is a bounded set, then*

$$\chi_{\tilde{\mathcal{T}}}(E) = \chi_{\mathcal{T}}(E) = \lambda(E).$$

Proof. Since $\mathcal{M}$ is countably generated, by Kasparov stabilization theorem, we have

$$\mathcal{M} \oplus H_{\mathcal{A}} =: \mathcal{M}_2 \cong H_{\mathcal{A}}.$$

Thus $\mathcal{M}$ is complemented in $\mathcal{M}_2$ and $\mathcal{M}_2 \cong H_{\mathcal{A}}$. Denote by $\chi^2_{\tilde{\mathcal{T}}}(E)$, $\chi^2_{\mathcal{T}}(E)$ and $\lambda_2(E)$ the corresponding measures of noncompactness with respect to $\mathcal{M}_2 \cong H_{\mathcal{A}}$. By Theorem 3.5 we have $\lambda_2(E) = \chi^2_{\tilde{\mathcal{T}}}(E)$.

On the other hand, by Proposition 3.16 we have $\lambda(E) = \lambda_2(E)$. Finally, apply Corollary 3.22 to obtain $\chi_{\mathcal{T}}(E) = \chi^2_{\mathcal{T}}(E)$. $\square$

**Corollary 3.24.** *Let $\mathcal{M}$ be a selfdual countably generated Hilbert module over a $W^*$-algebra $\mathcal{A}$, and let $E \subseteq \mathcal{M}$ be a bounded set. Then $\lambda(E) = \widehat{\lambda}(E)$, i.e. the $\lambda$ measure of noncompactness can be calculated via finitely generated projective submodules.*

Proof. Once again, by Kasparov stabilization theorem $\mathcal{M} \oplus H_{\mathcal{A}} \cong H_{\mathcal{A}}$, and $\mathcal{M}$ can be regarded as an orthogonal summand in $H_{\mathcal{A}}$. Since in this case both $\widehat{\lambda}$ and $\lambda$ does not depend on ambient module, the result follows. $\square$

### 3.3. Measures of noncompactness of operators.

*Definition 3.25.* Let $\mathcal{M}$ be a countably generated Hilbert module over a unital $C^*$-algebra $\mathcal{A}$ and let $T \in B^a(\mathcal{M})$ be an adjointable operator. Let $\alpha$, $\chi$, $I$ and $\lambda$ denote the corresponding measures of noncompactness with respect to Troitsky's topology. The functions $\alpha_o$, $\chi_o$, $I_o$, $\lambda_o : B^a(\mathcal{M}) \to [0, +\infty)$ defined by

$$\chi_o(T) = \chi(T(B_1)), \quad \alpha_o(T) = \alpha(T(B_1)),$$

$$I_o(T) = I(T(B_1)), \quad \lambda_o(T) = \lambda(T(B_1)),$$

where $B_1$ is the closed unit ball in $\mathcal{M}$ are called, respectively, Hausdorff, Kuratowski, Istrăţescu and $\lambda$ measure of noncompactness of the operator $T$ with respect to Troitsky's topology.

**Proposition 3.26.** *Let $\mathcal{M}$ be a countably generated Hilbert module over a $C^*$-algebra $\mathcal{A}$ and let $T \in B^a(\mathcal{M})$. Then*

$$\chi_0(T) = \inf\{k \geq 0 \mid \chi(T(E)) \leq k\chi(E) \textit{ for each bounded set } E\}.$$

Proof. The proof is, essentially, the same as in the case of metric spaces (see [18, Theorem 2.25].)

Indeed, let $k > 0$ be such that

$$\chi(T(E)) \leq k\chi(E), \tag{3.12}$$

holds for all bounded $E \subseteq \mathcal{M}$. For $E = B_1$ we obtain $\chi_0(T) = \chi(T(B_1)) \leq k\chi(B_1) = k$. Thus the infimum of such $k$ is greater or equal to $\chi_0(T)$.

For the other inequality, let $E$ be an arbitrary bounded set, let $k > \chi_0(T)$, and let $\eta > \chi(E)$. For any seminorm $p_\alpha$, there are two finite nets, the first $y_1$, $y_2$, $\dots$, $y_l$ makes a finite $k$ net for $B_1$, and the other $z_1$, $z_2$, $\dots$, $z_p$ makes a finite $\eta$ net for $E$. If $B(a;\delta)$ denote the ball with center at $a$ and radius $\delta$, then $E \subseteq \cup_{i=1}^p B(z_i;\eta)$ and hence

$$T(E) \subseteq \bigcup_{i=1}^p T(B(z_i;\eta)) = \bigcup_{i=1}^p (Tz_i + \eta T(B_1)).$$

On the other hand, $T(B_1) \subseteq \cup_{j=1}^l B(y_j;k)$ and therefore

$$T(E) \subseteq \bigcup_{i=1}^p \left( Tz_i + \eta \bigcup_{j=1}^l B(y_j;k) \right) = \bigcup_{i=1}^p \bigcup_{j=1}^l B(Tz_i + \eta y_j; \eta k).$$

Thus $\chi(T(E))$ does not exceed $\eta k$, which can be arbitrarily close to $k\chi(E)$. $\square$

**Proposition 3.27.** *Let $\mathcal{A}$ be an arbitrary $C^*$-algebra, let $T, S \in B^a(H_\mathcal{A})$ and let $\mu$ stand for any of the MNCs $\alpha, \chi, I$. Then*

a) *All $\alpha_o, \chi_o$ and $I_o$ are subadditive and positive homogeneous, i.e. there holds*

$$\mu_o(T+S) \leq \mu_o(T) + \mu_o(S), \quad \mu_o(cT) = c\mu(T) \quad \textit{for all } c > 0.$$

b) *The functions $\alpha_o, \chi_o, I_o$ and $\lambda_o$ are equivalent to each other, that is,*

$$\chi_o(T) \leq I_o(T) \leq \alpha_o(T) \leq 2\chi_o(T), \quad \chi_o(T) = \lambda_o(T).$$

c) *$\chi_o(T) \leq \|T\|$ and $\alpha_o(T), I_o(T) \leq 2\|T\|$.*

d) *Operator $T$ is $\mathcal{A}$-compact iff $\lambda_0(T) = 0$ iff $\mu_o(T) = 0$.*

e) *$\mu_o(T+K) = \mu_o(T)$, as well as $\lambda_o(T+K) = \lambda_o(T)$ for all $\mathcal{A}$-compact operators $K$.*

f) *There holds*

$$\chi_o(T) = \lambda_o(T) \leq I_o(T) \leq \alpha_o(T) \leq 2\chi_o(T).$$

PROOF. Parts a), b), c), and f) follow from the properties of $\alpha, \chi, I$ and $\lambda$. Part d) follows from part b) and Theorem 2.15. Part e) follows from part d) and the properties of $\alpha, \chi, I$ and $\lambda$. □

## 4. Concluding comment

*Remark 4.1.* Our results are restricted to countably generated modules. The proofs strongly relies on Kasparov stabilization theorem - a theorem without counterpart for uncountably generated modules. However, Troitsky's family (2.7), the measure of noncompactness $\lambda$, and Hausdorff measure $\chi$ are all defined in such a way that they make sense in uncountably generated modules.

Unfortunately, the basic result – the equivalence of $\mathcal{A}$-compactness with relatively compactness with respect to Troitsky's topology doesn't hold. Although one implication was proved in [7], a counterexample is constructed in [5] (see also, further development of this idea in [6]).

Hence, the first step toward uncountably generated modules would be to find an appropriate generalization of Troitsky's topology for them and to establish the mentioned equivalence. There are some results in this direction [8], but we can not say that we are close to the appropriate topology on uncountable modules that simultaneously generalizes the Troitsky's topology, and ensures the equivalence between $\mathcal{A}$-compactness and totally boundedness.

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