Surgery presentation of 3-dimensional small covers

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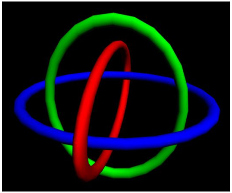
- By Lickorish and Wallace theorem (1962) every closed, orientable 3-manifold is obtained by surgery on some framed link $L \subset S^3$.
- Intriguing example: Surgery on Borromean link gives a torus T^3.



- Problem: Find a canonical surgery presentation of an orientable 3-dimensional small cover. Canonical means regarding to Kirby's moves on links.

3-dimensional small covers

- A Small cover $M \xrightarrow{\pi} P$ over a simple 3-dimensional convex polytope P is a closed 3-dimensional manifold with a locally standard $\mathbb{Z}_2^3$-action such that the orbit space $M/\mathbb{Z}_2^3$ is diffeomorphic to P as a manifold with corners.
- As a quotient space it is composed of eight copies of the polytope P glued together along their facets

  $$M = P \times \mathbb{Z}_2^3 / \sim, \qquad (p,a) \sim (q,b) \text{ iff } p = q,\ b - a \in G_p \subset \mathbb{Z}_2^3,$$

  where G_p is the isotropy subgroup of $p \in M$.
- The standard presentation $M = M(P,\Lambda)$ is given by the characteristic pair (P,Λ), where $\Lambda : \mathcal{F} \longrightarrow \mathbb{Z}_2^3$ is a coloring of the set of facets $\mathcal{F} = \{F_1, \dots, F_m\}$ of P satisfying the non-degeneracy condition

  $$\{\Lambda(F_i), \Lambda(F_j), \Lambda(F_k)\} \text{ is a linear bases for } \mathbb{Z}_2^3,\ v = F_i \cap F_j \cap F_k.$$

- Small covers $M(P,\Lambda)$ and $M(Q,\Lambda')$ are equivariantly diffeomorphic iff polytopes P,Q are combinatorially equivalent and there is a linear map $\theta \in Gl(\mathbb{Z}_2,3)$ such that $\Lambda' = \theta\Lambda$.
- $M = M(P,\Lambda)$ is orientable iff P is 4-colored.
- $M = M(P,\Lambda)$ is called linear model iff P is 3-colored. Linear models are uniquely determined by P up to reordering of colors $M = M(P)$. The polytope P is 3-colorable iff all its facets are evengons.

Example

$M(\Delta^3,\Lambda) = \mathbb{R}P^3$
$M(\square^3) = T^3$

Some chronology

- Already in the seminal paper of Davis and Januszkiewicz there is a remark about geometrization of 3-dimensional small covers.
- The construction of small covers out of some elementary pieces using a finite set of local moves was studied by Izmestiev (2001), L*ü*, Yu (2009), Kuroki (2010) and Nishimura (2012).
- Buchstaber, Erokhovets, Masuda, Panov and Park (2017) proved the cohomological rigidity of small covers over Pogorelov class.
- Erokhovets (2020) built an explicit decomposition of 3-dimensional small covers into geometric parts.

- For a facet $F \in \mathcal{F}$ of P the facet submanifold $M_F = \pi^{-1}(F)$ is a 2-dimensional small cover $M_F = M(F, \Lambda_F)$, where $\Lambda_F : \{F' \in \mathcal{F} | F \cap F' \neq \emptyset\} \longrightarrow \mathbb{Z}_2^2$ is the coloring induced by projection $\rho_F : \mathbb{Z}_2^3 \to \mathbb{Z}_2^3 / \langle \Lambda(F) \rangle$

$$\Lambda_F(F \cap F') = \rho_F(\Lambda(F')),\ F \cap F' \neq \emptyset.$$

  For linear models it is simply given by $\Lambda_F(F \cap F') = \Lambda(F')$.
- The facet submanifold $M_F = M(F, \Lambda_F)$ is an orientable surface $M_F = S_g$ of genus g iff the facet F is a 2-colored $2g+2$-gon $F = P_{2g+2}$. Thus for linear models $M = M(P)$ all facet submanifolds $M_F,\ F \in \mathcal{F}$ are oriented surfaces.
- Each edge uv of P induces a circle $\widehat{uv} \subset M$.

Haken manifolds

- Closed, orientable 3-manifold M is a Haken manifold (or sufficiently large) iff it is irreducible (every embedded 2-sphere bounds a ball) and contains incompressible surface $S_g \subset M$ of positive genus $g > 0$. Incompressibility is equivalent by Dehn's Loop lemma to π_1-injectivity.
- Closed, orientable, irreducible 3-manifold with $\beta_1(M) \geq 1$ is a Haken manifold.
- Haken manifolds are aspherical, i.e. all higher homotopy groups are trivial. It follows from the Sphere theorem by Papakyriakopoulos, 1957. Therefore Haken manifolds are $K(\pi, 1)$ spaces.
- Waldhausen (1968) shown that fundamental group determines diffeomorphism types of Haken manifolds.
- Thurston proved that atoroidal Haken manifolds are hyperbolic (hyperbolic manifolds need not to be Haken).

π_1-injectivity

Combinatorial criteria for π_1-injectivity of face submanifolds [1]

Theorem (L.Wu, L.Yu, 2021)

For a small cover $M = M(P, \Lambda)$ and a proper face $f \subset P$ the facial submanifold $M_f = \pi^{-1}(f)$ is π_1-injective iff the following combinatorial condition holds:
For any adjoint facets F, F' normal to f we have $f \cap F \cap F' \neq \emptyset$.

Combining with the result of Davis, Januszkiewicz and Scott [2], we obtain the equivalency of conditions for small covers $M(P, \Lambda)$

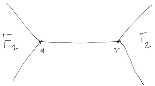
- M is aspherical,
- P is flag polytope,
- All facial submanifolds M_F are π_1-injective.

[1] *Fundamental groups of small covers revisited*, IMRN **10** (2021), 7262–7298.

[2] *Nonpositive curvature of blow-ups*, Selecta Math. **4** (1998), 491–547.

What does it mean for 3-dimensional small covers?

- The facial surface M_F of a 3-dimensional small cover $M = M(P, \Lambda)$ is π_1-injective iff a facet $F \subset P$ is not contained in any 3-belt. There always exists such a facet. If there is a 2-colored facet which is an evengon then M is a Haken manifold.
- A circle $\widehat{uv} \subset M$ generates an infinite cyclic subgroup $\mathbb{Z} \subset \pi_1(M)$ iff the normal facets F_1 and F_2 are not adjacent.



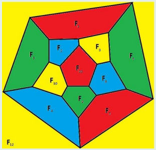
- Therefore the fundamental group of M is infinite for $P \neq \Delta^3$.

- An orientable 3-dimensional small cover $M(P,\Lambda)$ over a flag polytope $P \neq \Delta^3$ is a closed, oriented, irreducible 3-manifold with the infinite fundamental group.
- Virtual Haken conjecture (Waldhausen, 1968.) states that all such manifolds have a finite cover which is Haken.
- Question: Are all orientable 3-dimensional small covers Haken manifolds? Prove that they are virtually Haken (the negative answer would disprove the conjecture, which is too ambitious).
- At least the linear model $M(P)$ over a flag polytope P is a Haken manifold. Therefore its diffeomorphism type is determined by the fundamental group.

We use the Suciu-Trevisan formula for rational Betti numbers of small covers to prove: If P is a flag 4-colored polytope and there is a 2-colored $2k$-belt on P then the 3-dimensional small cover $M(P, \Lambda)$ is a Haken manifold.

Example (dodecahedral space)

A coloring of the dodecahedron [a] that produces an irreducible, orientable small cover which is a hyperbolic manifold with rational Betti numbers $\beta_1 = \beta_2 = 0$ (calculated by the Suciu-Trevisan formula).



[a] the figure is borrowed from Buchstaber, Erokhovets, *Fullerenes, Polytopes and Toric topology*, 2016.

The fundamental group $\pi_1(M(P,\Lambda))$

- The right-angled Coxeter group W_P of a polytope P is defined by $W_P = \langle s_F, F \in \mathcal{F} | s_F^2 = 1, s_F s_{F'} = s_{F'} s_F, F \cap F' \neq \emptyset \rangle$.
- The universal covering space for all simple covers over P is the quotient space

$$\mathcal{L}_P = P \times W_P / \sim, \quad (p, a) \sim (q, b) \text{ iff } p = q, ab^{-1} \in \langle s_F, p \in F \rangle.$$

- From the homotopy exact sequence of the Borel fibration for $M = M(P, \Lambda)$

$$M \longrightarrow E\mathbb{Z}_2^3 \times_{\mathbb{Z}_2^3} M \longrightarrow B\mathbb{Z}_2^3$$

we have the exact sequence

$$1 \to \pi_1(M) \longrightarrow W_P \xrightarrow{\phi} \mathbb{Z}_2^3 \to 1,$$

where ϕ is given by $\phi(s_F) = \Lambda(F),\ F \in \mathcal{F}$. Thus

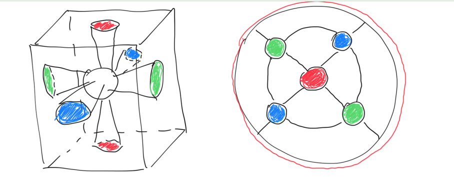
$$\pi_1(M) \cong \ker \phi.$$

Heegaard diagram

- The more combinatorial description of the fundamental group of a 3-dimensional small cover can be derived from Heegaard diagrams.
- The decomposition of a closed, orientable 3-manifold M into two handlebodies of the same genus is called a Heegaard splitting. The smallest number of handles $\gamma(M)$ is called the Heegard genus of M.
- A Heegaard diagram on the handlebody H_g is a set of disjoint, simple, closed curves $\{\lambda_1, \ldots, \lambda_g\}$ such that cutting the boundary surface ∂H_g along these curves produces a sphere with $2g$ holes. Given such a set we obtain a closed, orientable 3-manifolds by attaching 2-handles along curves in the set.

- A Heegard diagram is described as a system of arcs on the sphere with $2g$ holes obtained by cutting the surface ∂H_g along the standard meridians set $\{\mu_1, \ldots, \mu_g\}$. Any closed, oriented 3-manifold has a Heegaard diagram.

Example (3-torus)



- The fundamental group of a 3-manifold can be easily read off the Heegaard diagram

$$\pi_1(M) = \langle s_1, \dots, s_g | R_1, \dots, R_g \rangle,$$

where $s_1, \dots, s_g$ are cores of handles and $R_1, \dots, R_g$ are words obtained according to the set of curves $\{\lambda_1, \dots, \lambda_g\}$.
- The rank $r(M)$ of the group $\pi_1(M^3)$ is its minimal number of generators. It holds $r(M) \leq \gamma(M)$. There are examples of the strong inequality.
- We have the upper bound $\gamma \leq h_1 = f - 3$ since there is a Morse function on M with $h_1 = f - 3$ critical points of index 1. On the other hand $f - 3 \leq r$ since there is an epimorphism $\pi_1(M) \to H_1(M, \mathbb{Z}) \to H_1(M, \mathbb{Z}_2)$ and $\beta_1^{\mathbb{Z}_2}(M) = h_1 = f - 3$. Therefore $\gamma(M) = f - 3$.

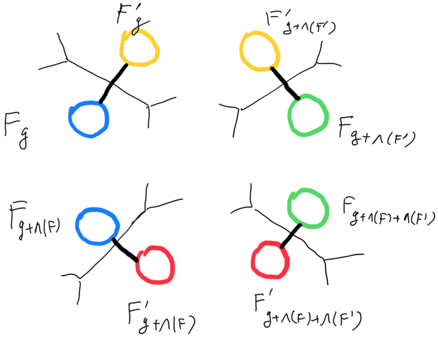
- A Heegaard splitting of a small cover $M = M(P, \Lambda)$ is made of thickened edges of polytopes $P_g, g \in \mathbb{Z}_2^3$ and their complements glued according to the coloring Λ

$$F_g \sim F_{g+\Lambda(F)},\ F \in \mathcal{F}, g \in \mathbb{Z}_2^3.$$

- The Wu and Yu presentation of $\pi_1(M, v)$:
generators: $\beta_{F,g}, F \in \mathcal{F}, g \in \mathbb{Z}_2^3$
relations:

$$\beta_{F,g}\beta_{F,g+\Lambda(F)} = 1, F \in \mathcal{F}, g \in \mathbb{Z}_2^3,$$
$$\beta_{F,g}\beta_{F',g+\Lambda(F)} = \beta_{F',g}\beta_{F,g+\Lambda(F')}, F \cap F' \neq \emptyset,$$
$$\beta_{F,g} = 1, v \in F.$$

The middle relation is a consequence of the 4-cycle of identifications of facets, which gives the following part of the Heegaard diagram of M

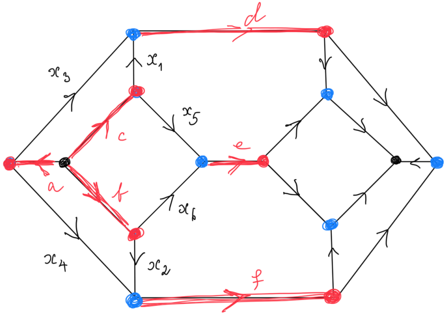


$$\beta_{F,g}\,\beta_{F',g+\Lambda(F)}\,\beta^{-1}_{F,g+\Lambda(F')}\,\beta^{-1}_{F',g}=1$$

Morse theoretic approach

- Let $\mu : M(P,\Lambda) \to P$ be a moment map and $l : P \to \mathbb{R}$ a generic linear map. Then $l\mu : M(P,\Lambda) \to \mathbb{R}$ is a Morse function.
- The critical points of $l\mu$ are the vertices of polytope P and the Morse index of a critical point $v \in M$ equals the index $\mathrm{ind}(v)$ of the the vertex $v \in P$ defined as the number of incoming edges of P under linear functional l.
- The 1-skeleton of P is turned into an oriented graph. Let v_0 be the minimal vertex.
- The generators of the fundamental group $\pi_1(M, v_0)$ are in correspondence with critical vertices of index $\mathrm{ind}(v) = 1$ and relations are read off facets with maximal vertices of index $\mathrm{ind}(v) = 2$.

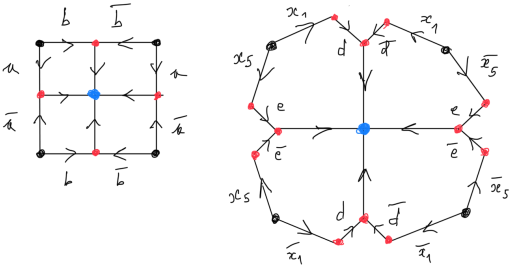
Example $T^3 \sharp T^3$



$$\alpha = a\overline{a}^{-1}, \beta = b\overline{b}^{-1}, \gamma = c\overline{c}^{-1},$$

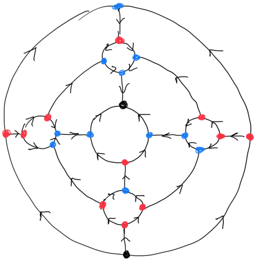
$$\delta = cx_1 d\overline{d}^{-1}x_1^{-1}c^{-1}, \epsilon = cx_5 e\overline{e}^{-1}x_5^{-1}c^{-1}, \varphi = bx_2 f\overline{f}^{-1}x_5^{-1}b^{-1}.$$

$\pi_1(T^3 \sharp T^3) = \mathbb{Z}^3 * \mathbb{Z}^3$



$$\alpha\beta\alpha^{-1}\beta^{-1} = 1, \delta\beta^{-1}\epsilon\beta\alpha^{-1}\delta^{-1}\alpha\epsilon^{-1} = 1.$$

Permutohedral space



$$BC = CB, DE = ED, DB^{-1}FBA^{-1}D^{-1}AF^{-1} = 1, GA^{-1}EAC^{-1}G^{-1}CE^{-1} = 1,$$
$$HDB^{-1}H^{-1}BD^{-1} = 1, IEC^{-1}I^{-1}CE^{-1} = 1, JAF^{-1}HFA^{-1}E^{-1}J^{-1}EH^{-1} = 1,$$
$$JAG^{-1}IGA^{-1}D^{-1}J^{-1}DI^{-1} = 1, HC^{-1}KCF^{-1}H^{-1}FK^{-1} = 1,$$
$$KG^{-1}IGB^{-1}K^{-1}BI^{-1} = 1, HC^{-1}IB^{-1}CH^{-1}BI^{-1} = 1.$$