On a class of highly non-immersable manifolds

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What is a highly non-immersable manifold?

Problem (The Embedding Problem:)

Is X a subspace of Y?

Definition

An embedding $i : X \hookrightarrow Y$ is a continuous injective map which is a homeomorphism onto the image $i(X) \subset Y$.

In smooth category: X and Y are smooth manifolds, all maps are smooth.

Definition

An immersion is a smooth map $i : X \looparrowright Y$ such that the differential df_x is injective at any $x \in X$.

It is an embedding locally, but it allows self-intersections globally.

The Foundational Results:

Theorem (Hassler Whitney, 1936. and 1944.)

- *weak:* $X^n \hookrightarrow \mathbb{R}^{2n+1}$, $X^n \looparrowright \mathbb{R}^{2n}$,
- *strong:* $X^n \hookrightarrow \mathbb{R}^{2n}$, $X^n \looparrowright \mathbb{R}^{2n-1}$.

How can it be improved?

$\alpha(n)$ =the number of digits 1 in the binary expansion of n.

$\alpha(6) = \alpha(110)_2 = 2, \alpha(7) = \alpha(111)_2 = 3, \alpha(8) = \alpha(1000)_2 = 1$

Theorem (Immersion Theorem, Ralph Cohen, 1985.)

$X^n \looparrowright \mathbb{R}^{2n-\alpha(n)}$.

Definition

X^n is highly non-immersable manifold if

$$X^n \not\looparrowright \mathbb{R}^{2n-\alpha(n)-1}.$$

What are obstructions for immersions?

Definition

Stiefel-Whitney classes: $w_i \in H^i(X^n, \mathbb{Z}_2), i = 1, \ldots, n$.
They measure how the tangent bundle TX twists along the manifold X.

If $X \looparrowright \mathbb{R}^N$ then $TX \oplus \nu_X \cong \epsilon_N$. All normal bundles of immersions are stable equivalent.
Dual Stiefel-Whitney classes:
$(1 + w_1 + \cdots + w_n)(1 + \overline{w}_1 + \cdots + \overline{w}_n) = 1$.

Theorem

If $\overline{w}_k \neq 0$ *then* $X \not\looparrowright \mathbb{R}^{n+k-1}$.

Find as many possible classes of n-dimensional manifolds with the property $\overline{w}_{n-\alpha(n)} \neq 0$.

Number-theoretic fact: The parity of binomial coefficients: $\binom{n}{k}$ is odd if and only if 1's in $(k)_2$ are 1's in $(n)_2$.

Example (from the Milnor-Stasheff book)

$$H^*(\mathbb{R}P^n, \mathbb{Z}_2) = \mathbb{Z}_2[x]/(x^{n+1}),$$

$$w(\mathbb{R}P^n) = (1+x)^{n+1},$$

$$\overline{w}_k = \binom{n+k}{k} x^k, k = 1, \ldots, n.$$

If $n = 2^p$ then $\overline{w}_{n-1} \neq 0$. Conclusion: $\mathbb{R}P^{2^p}$ is highly non-immersable.

In any dimension there are highly non-immersable manifolds:

Example (product of real projective spaces)

$$X^n = \mathbb{R}P^{2^{j_1}} \times \cdots \times \mathbb{R}P^{2^{j_m}}, \quad n = 2^{j_1} + \ldots + 2^{j_m}.$$

$$\overline{w}_{n-\alpha(n)}(X) = \overline{w}_{2^{j_1}-1}(\mathbb{R}P^{2^{j_1}}) \cdots \overline{w}_{2^{j_m}-1}(RP^{2^{j_m}}) \neq 0.$$

Toric geometry

Toric topology: manifolds with toric actions $T^n \curvearrowright X$

Definition (M. Demazure, 1970.)

A toric manifold X of the complex dimension n is a smooth, compact, complex algebraic variety that contains the complex algebraic torus $T_{\mathbb{C}} = (\mathbb{C}^*)^n \subset X$ as an open, dance subset such that the action $T_{\mathbb{C}} \curvearrowright T_{\mathbb{C}}$ extends smoothly

$$T_{\mathbb{C}} \curvearrowright X.$$

The real locus is the conjugation invariant real submanifold

$$X^{\mathbb{R}} \subset X.$$

Theorem (Fundamental theorem of toric geometry)

There is one-to-one correspondence between toric manifolds and complete, regular fans

$$\Sigma \longleftrightarrow X_{\Sigma}.$$

Definition

A fan in $\mathbb{R}^n$ is a finite collection of cones $\Sigma = \{\sigma\}$ such that

- if $\tau \subset \sigma \in \Sigma$ then $\tau \in \Sigma$,
- $\sigma \cap \tau \in \Sigma$ for $\tau, \sigma \in \Sigma$.

Σ is complete if $\bigcup \Sigma = \mathbb{R}^n$ and regular if any cone σ is generated by a part of a basis of the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

A complete and regular fan Σ determines a regular star-shaped simplicial sphere combinatorially equivalent to

$$\mathcal{K}_{\Sigma} = \Sigma \cap S^{n-1}.$$

Theorem

If $\Sigma = \mathcal{N}_P$ is a normal fan of a simple convex polytope P then $X_\Sigma \hookrightarrow \mathbb{C}P^N$ is a projective toric manifold.

- Not all simplicial spheres are star-shaped.
- Not all star-shaped spheres are polytopal.

Bier's spheres

Definition

A Bier's sphere $\mathrm{Bier}(\mathcal{K})$ of a simplicial complex $\mathcal{K}$ on the set of vertices $[m] = \{1, 2, \ldots, m\}$ is

$$\mathrm{Bier}(\mathcal{K}) = \mathcal{K} *_{\Delta} \mathcal{K}^{\vee},$$

where

$$\mathcal{K}^{\vee} = \{\tau \subset [m] \mid \tau^{c} \notin \mathcal{K}\}$$

is the Alexander dual of $\mathcal{K}$.

Theorem (I. Limonchenko, M. Timotijević, R. Živaljević, 2025.)

$\mathrm{Bier}(\mathcal{K})$ *has a canonical star-shaped realization.*

Canonical toric manifolds $X_{\mathcal{K}}$

To a simplicial complex on $[m]$ corresponds a canonical toric manifold of the complex dimension $m-1$ and its real locus the canonical real toric manifold

$$\mathcal{K} \longrightarrow X_{\mathcal{K}}, X_{\mathcal{K}}^{\mathbb{R}}.$$

The integer cohomology ring of the canonical toric manifold $X_{\mathcal{K}}$ is determined by the Danilov-Jurkiewicz theorem as the quotient ring

$$H^*(X_{\mathcal{K}};\mathbb{Z}) \cong \mathbb{Z}[\mathrm{Bier}(\mathcal{K})]/\mathcal{J} = \mathbb{Z}[x_1,\dots,x_m,x'_1,\dots,x'_m]/_{\mathcal{I}+\mathcal{J}},$$

where $\mathcal{I}$ is the Stanley-Reisner ideal of the complex $\mathrm{Bier}(\mathcal{K})$, which has the following form:

$$\mathcal{I} = \mathcal{I}_{\mathcal{K}} + \mathcal{I}_{\mathcal{K}^\vee} + \langle x_i x'_i, i = 1,\dots,m\rangle$$

with $\mathcal{I}_{\mathcal{K}}$ and $\mathcal{I}_{\mathcal{K}^\vee}$ being the Stanley–Reisner ideals and $\mathcal{J}$ is the ideal of linear dependencies

$$x_i - x'_i = x_m - x'_m, \quad i = 1,\dots,m-1.$$

The total Stiefel–Whitney class of the canonical real toric manifold is given by

$$w(X_{\mathcal{K}}^{\mathbb{R}}) = \prod_{i=1}^{m}(1 + x_i + x_i') = (1+u)^m,$$

which gives

$$\overline{w}_k(X_{\mathcal{K}}^{\mathbb{R}}) = \binom{m+k-1}{k} u^k, \quad k = 1, \ldots, m-1.$$

It follows

$$k_{\max} = \max\{k : \overline{w}_k(X_{\mathcal{K}}^{\mathbb{R}}) \neq 0\} = (2^p - 1) - (m-1),$$

for

$$2^{p-1} < m \leq 2^p.$$

Theorem

The canonical real toric manifold $X_{\mathcal{K}}^{\mathbb{R}}$ *is highly non-immersable*

$$X_{\mathcal{K}}^{\mathbb{R}} \not\looparrowright \mathbb{R}^{2m-4}$$

for any simplicial complex $\mathcal{K} \neq \Delta_{[m]}$ *on* $m = 2^{p-1} + 1$ *vertices.*

Theorem

Let $m-1 = 2^{j_1} + \cdots + 2^{j_k}$ *and* $\mathcal{K}_i \neq \Delta_{[2^{j_i}+1]}$, $i = 1, \ldots, k$ *be simplicial complexes on* $2^{j_i}+1$, $i = 1, \ldots, k$ *vertices respectively. The following manifold is highly non-immersable*

$$X^{\mathbb{R}}_{\mathcal{K}_1} \times \cdots \times X^{\mathbb{R}}_{\mathcal{K}_k} \not\looparrowright \mathbb{R}^{2m-k-3}.$$

V. Grujić, I. Limonchenko, *Characteristic numbers of canonical toric manifolds and their applications*, arXiv:2604.02138