1. Dokazati da je funkcija $p: \mathbb{N} \to \mathbb{N}$ data sa $p(n) =$ „n-ti prost broj" primitivno rekurzivna.

$p(0) = 2$

$p(1) = 3$ $\qquad p(n+1) \leq p(n)! + 1$

$p(2) = 5$ $\qquad$ Ako bismo pretpostavili da je

$\vdots$ $\qquad p(n+1) > p(n)! + 1$, tada prosti faktori od $p(n)! + 1$ mogu biti samo $p(0), p(1), \dots, p(n)$. Međutim,

$p(0), p(1), \dots, p(n) \mid p(n)!$

$\Rightarrow p(0), p(1), \dots, p(n) \nmid p(n)! + 1$

$\Rightarrow p(n)! + 1$ je prost broj $\lightning$

Dakle, $p(n+1) \leq p(n)! + 1$

$p(0) = 2$

$p(n+1) = (\mu q \leq p(n)! + 1)(q \in \mathbb{P} \wedge q > p(n))$

Funkcija $n \xrightarrow{g} n!$ jeste primitivno rekurzivna

$g(0) = 0! = 1$

$g(n+1) = (n+1)! = n! \cdot (n+1) = g(n) \cdot S(n)$

2. Dokazati da je funkcija

$$(x)_y = \begin{cases} 0 & x = 0 \\ \text{najveći stepen prostog broja } p(y) \text{ koji deli } x & \\ \Leftrightarrow \text{stepen od } p(y) \text{ koji se javlja u kanonskoj faktorizaciji od } x & \end{cases}$$

$x = 20 = 2^2 \cdot 5$

$(x)_0 = 2$ $\qquad$ $(x)_y$ - takav broj da

$(x)_1 = 0$ $\qquad$ $p(y)^{(x)_y} \mid x$

$(x)_2 = 1$

$\qquad$ $p(y)^{(x)_y + 1} \nmid x$

$$(x)_y = (\mu_z \leq x)\left((p(y)^z, x) \in | \wedge (p(y)^{z+1}, x) \notin | \right) \dot{-} \chi_{\{0\}}(x)$$

$$x = p(y)^z \cdot \text{nešto}$$

$$p(y)^z \leq x \quad / \log_{p(y)}$$

$$z \leq \log_{p(y)} x \leq x \Rightarrow z \leq x$$

$$\ulcorner f(x) = x - \log_{p(y)} x \geq 0 \lrcorner$$

3. Dokazati da je funkcija $f: \mathbb{N} \to \mathbb{N}$ data sa:

$f(0) = 1$

$f(1) = 2$

$f(2) = 5$

$f(n+3) = f(n) + f(n+1) + f(n+2)$

primitivno rekurzivna.

Kreiramo funkciju koja čuva informaciju o tri uzastopna člana niza f.

$$g(n) = 2^{f(n)} \cdot 3^{f(n+1)} \cdot 5^{f(n+2)}$$

Ako $g \in PR \Rightarrow f \in PR$ jer $f(n) = (g(n))_0$

$$g(0) = 2^{f(0)} \cdot 3^{f(1)} \cdot 5^{f(2)} = 2^1 \cdot 3^2 \cdot 5^5 \in \mathbb{N}$$

$$g(n+1) = 2^{f(n+1)} \cdot 3^{f(n+2)} \cdot 5^{f(n+3)} =$$

$$2^{(g(n))_1} \cdot 3^{(g(n))_2} \cdot 5^{f(n) + f(n+1) + f(n+2)} =$$

$$2^{(g(n))_1} \cdot 3^{(g(n))_2} \cdot 5^{(g(n))_0 + (g(n))_1 + (g(n))_2}$$

Za vežbu dokazati:

Ako je $a \in \mathbb{N}$, tada je $n \mapsto a^n \in PR$

4. Dokazati da je Fibonačijev niz $\in PR$

$F(0)=1$

$F(1)=1$

$F(n+1) = F(n) + F(n-1)$

5. Neka je $h(\overline{m}, n, p) \in PR$ i $f(\overline{m}, n+1) = h(\overline{m}, n, \widehat{f}(\overline{m}, n))$, gde je $\widehat{f}(\overline{m}, n) = p(0)^{f(\overline{m},0)} \cdot p(1)^{f(\overline{m},1)} \cdot \ldots \cdot p(n)^{f(\overline{m},n)}$ i $f(\overline{m}, 0) \in PR$ po $\overline{m}$. Dokazati da su f i $\widehat{f}$ $\in PR$.

Dokažimo prvo da je $\widehat{f} \in PR$

$$\widehat{f}(\overline{m}, 0) = p(0)^{f(\overline{m},0)} \quad , \text{ali } f(\overline{m},0) \in PR$$
$$= 2^{f(\overline{m},0)} \in PR \text{ po } \overline{m}$$

$$\widehat{f}(\overline{m}, n+1) = \underbrace{p(0)^{f(\overline{m},0)} \cdot \ldots \cdot p(n)^{f(\overline{m},n)}}_{\widehat{f}(\overline{m},n)} \cdot p(n+1)^{f(\overline{m},n+1)}$$
$$= \widehat{f}(\overline{m}, n) \cdot p(n+1)^{f(\overline{m},n)} =$$
$$\widehat{f}(\overline{m}, n) \cdot p(n+1)^{h(\overline{m}, n, \widehat{f}(\overline{m}, n))}$$

$\Rightarrow \widehat{f} \in PR$

$f(\overline{m}, n) = (\widehat{f}(\overline{m}, n))_n \Rightarrow f \in PR$

II način:

$$\left.\begin{array}{l} f(\overline{m},0) \in PR \\ f(\overline{m},n+1) = h(\overline{m},n,f(\overline{m},n)) \end{array}\right\} \Rightarrow f \in PR$$

6. Neka su f i g funkcije definisane sa:

$$f(0) = g(0) = 1$$
$$f(x+1) = g(x) \dot{-} f(x)$$
$$g(x+1) = g(x) + f(x)$$

Dokazati da $f, g \in PR$.

Uvodimo pomoćnu funkciju $h: \mathbb{N} \to \mathbb{N}$ datu sa:

$$h(x) = 2^{f(x)} \cdot 3^{g(x)}$$

Ako $h \in PR \Rightarrow \left.\begin{array}{l} f(x) = (h(x))_0 \\ g(x) = (h(x))_1 \end{array}\right\} \Rightarrow f, g \in PR$

$$\left.\begin{array}{l} h(0) = 2^{f(0)} \cdot 3^{g(0)} = 2^1 \cdot 3^1 = 6 \\ h(x+1) = 2^{f(x+1)} \cdot 3^{g(x+1)} = 2^{g(x) \dot{-} f(x)} \cdot 3^{g(x)+f(x)} \\ \qquad = 2^{(h(x))_1 \dot{-} (h(x))_0} \cdot 3^{(h(x))_1 + (h(x))_0} \end{array}\right\} \Rightarrow h \in PR$$

7. Funkcija $f: \mathbb{N} \to \mathbb{N}$ definisana sa:

$$f(0) = 1$$
$$f(x+1) = f\left(\left[\frac{x}{2}\right]\right) + x$$

Dokazati da $f \in PR$

Definišemo pomoćnu funkciju

$$g(x) = 2^{f(0)} \cdot 3^{f(1)} \cdots p(x)^{f(x)}$$ $\leftarrow$ čuvamo informacije o početnih $x+1$ vrednosti funkcije f

Ako $g \in PR \Rightarrow f(x) = (g(x))_x \Rightarrow f \in PR$

Stoga, dokazujemo da je $g \in PR$:

$$\underline{g(0) =} 2^{f(0)} = 2^1 = 2 \in \mathbb{N}$$

$$g(x+1) = 2^{f(0)} \cdot \ldots \cdot p(x)^{f(x)} \, p(x+1)^{f(x+1)} = g(x) \cdot p(x+1)^{f(x+1)}$$

$$= g(x) \, p(s(x))^{f([\frac{x}{2}]) + x}$$

Ali $f([\frac{x}{2}])$ je eksponent uz $p([\frac{x}{2}])$ u faktorizaciji $g(x) \Rightarrow f([\frac{x}{2}]) = (g(x))_{[\frac{x}{2}]}$

$$\Rightarrow \underline{g(x+1) =} g(x) \cdot p(s(x))^{(g(x))_{[\frac{x}{2}]} + x}$$

Dokazali smo da $div_2 \in PR$

$\Rightarrow g \in PR$.

8. Dokazati da je funkcija $x \mapsto \binom{x}{2} \in PR$

$$\binom{0}{2} = \binom{1}{2} = 0 \qquad \binom{x}{2} = \frac{x!}{2!(x-2)!} = \frac{x(x-1)}{2}$$

$$\binom{x}{2} = \frac{x(x-1)}{2} = div_2(x \cdot (\underbrace{x \dot{-} 1}_{\delta(x)})) \Rightarrow \binom{x}{2} \in PR$$

II način: preko primitivno rekurzivne šeme:

$$\left.\begin{aligned} &\binom{0}{2} = 0 \\ &\binom{x+1}{2} = \frac{(x+1) \cdot x}{2} = \frac{(x-1+2) \cdot x}{2} = \frac{x \cdot (x-1) + 2x}{2} = \\ &\frac{x(x-1)}{2} + x = \binom{x}{2} + \underset{= \binom{x}{1}}{x} \end{aligned}\right\} \Rightarrow x \mapsto \binom{x}{2} \in PR$$

Def: Kantorova funkcija $K : \mathbb{N}^2 \to \mathbb{N}$ je zadata sa $K(x,y) = \binom{x+y+1}{2} + x = \frac{(x+y+1)(x+y)}{2} + x$

Stav: $\mathcal{K} \in PR$

Stav: $\mathcal{K}$ je bijekcija

dokaz: „1-1": $\mathcal{K}(x_1,y_1) \overset{(*)}{=} \mathcal{K}(x_2,y_2)$

Prvo dokazujemo da je $x_1+y_1 = x_2+y_2$

Pps da $x_1+y_1 \neq x_2+y_2$, tj. BUO da je $x_1+y_1 < x_2+y_2$

$$\mathcal{K}(x_2,y_2) = \binom{x_2+y_2+1}{2} + \underset{\geq 0}{x_2} \geq \binom{x_2+y_2+1}{2} \geq$$

$\big[\, x_2+y_2 > x_1+y_1 \Leftrightarrow x_2+y_2 \geq x_1+y_1+1 \,\big]$

$$\binom{x_1+y_1+1+1}{2} = \underbrace{\binom{x_1+y_1+1}{2} + x_1+y_1}_{= \mathcal{K}(x_1,y_1)}+1$$

$$> \mathcal{K}(x_1,y_1)$$

↯

Dakle, $x_1+y_1 \overset{(☺)}{=} x_2+y_2 \Rightarrow x_1+y_1+1 = x_2+y_2+1$

$$\Rightarrow \binom{x_1+y_1+1}{2} = \binom{x_2+y_2+1}{2}$$

(*) daje: $\cancel{\binom{x_1+y_1+1}{2}} + x_1 = \cancel{\binom{x_2+y_2+1}{2}} + x_2$

$\Rightarrow x_1 = x_2$

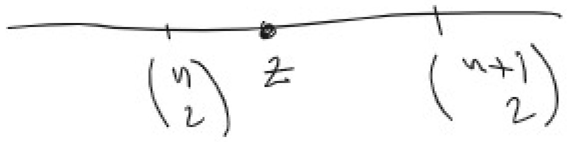
(☺) daje $y_1 = y_2$

$\Rightarrow (x_1,y_1) = (x_2,y_2)$

$\Rightarrow \mathcal{K}$ jeste „1-1".

„na": Za proizvoljno $z \in \mathbb{N}$ postoji $(x,y) \in \mathbb{N}^2$ takav da je $\mathcal{K}(x,y) = z$

$$\binom{x+y+1}{2} + x = z$$



Tražimo $n \in \mathbb{N}$ tako da je

$$\binom{n}{2} \leq z < \binom{n+1}{2}$$

$$x = z - \binom{n}{2} \geq 0$$

$$n = x + y + 1$$

$$y = n - (x+1) \geq 0$$

Ako je $z = 0$ $\quad x = 0, y = 0$

Ako je $z = 1$

$$\binom{0}{2} = 0 \qquad \binom{2}{2} \leq 1 < \binom{3}{2}$$

$$\binom{1}{2} = 0 \qquad \left.\begin{aligned} x &= 1 - \binom{2}{2} = 1 - 1 = 0 \\ y &= 2 - (0+1) = 1 \end{aligned}\right\} \Rightarrow (0,1) \mapsto 1$$

$$\binom{2}{2} = 1 \qquad \binom{3}{2} = 3$$

Ako je $z = 2$:

$$\binom{2}{2} \leq 2 < \binom{3}{2}$$

$$\Rightarrow \left.\begin{aligned} x &= 2 - \binom{2}{2} = 2 - 1 = 1 \\ y &= 2 - (1+1) = 2 - 2 = 0 \end{aligned}\right\} \Rightarrow (1,0) \mapsto 2$$

$\vdots$

miro