



Belonging of Gel'fand integral of positive operator valued functions to separable ideals of compact operators on Hilbert space

Mihailo Krstić¹ · Matija Milović¹ · Stefan Milošević¹

Received: 22 September 2022 / Accepted: 10 November 2022

© The Author(s), under exclusive licence to Springer Nature Switzerland AG 2022

Abstract

We provide some sufficient and necessary conditions for weak* integral of operator valued function to belong to ideals of compact operators. Also, implications of such inclusions are considered, specially in the case of compact operators. Applications are given as well.

Keywords Weak* integral · Operator valued measure · Unitarily invariant norm · Symmetrically norming function

Mathematics Subject Classification Primary 47A30 · 46G10; Secondary 47A65 · 47B10 · 47B15 · 46B20

1 Introduction

Throughout the paper, we will denote by $\mathcal{B}(\mathcal{H})$ and $\mathcal{C}_\infty(\mathcal{H})$ spaces of all bounded and all compact linear operators respectively acting on a separable, infinite-dimensional, complex Hilbert space $\mathcal{H}$. For a symmetric norming (s.n.) function Φ defined on sequences, we will denote by $\|\cdot\|_\Phi$ symmetric norm or a unitarily invariant (u.i.) norm on operators. The latter is defined by $\|X\|_\Phi \stackrel{\text{def}}{=} \Phi((s_n(X))_{n=1}^\infty)$, where $s_1(X) \geq s_2(X) \geq \dots$ are the singular values of operator X . Also, such norms will be denoted by the symbol $|||\cdot|||$. Any u.i. norm $|||\cdot|||$ is defined on a naturally associated norm ideal

✉ Stefan Milošević
stefanm@matf.bg.ac.rs

Mihailo Krstić
mihailo_krstic@matf.bg.ac.rs

Matija Milović
matija_milovic@matf.bg.ac.rs

¹ Department of Mathematics, University of Belgrade, Studentski trg 16, P.O.Box 550, 11000 Belgrade, Serbia

$\mathcal{C}_{|||\cdot|||}(\mathcal{H})$ of $\mathcal{C}_\infty(\mathcal{H})$ and satisfies the invariance property $|||UXV||| = |||X|||$ for all $X \in \mathcal{C}_{|||\cdot|||}(\mathcal{H})$ and for all unitary operators U, V on $\mathcal{H}$.

For a s.n. function Φ and $p \geq 1$ we will denote by $\Phi^{(p)}$ its (degree) p modification, defined by

$$\Phi^{(p)}((z_n)_{n=1}^\infty) \stackrel{\text{def}}{=} \sqrt[p]{\Phi((|z_n|^p)_{n=1}^\infty)},$$

whose associated ideal consists of all complex sequences $z = (z_n)_{n=1}^\infty$ that satisfy $(|z_n|^p)_{n=1}^\infty \in \ell_\Phi$. Modification $\Phi^{(p)}$ is s.n. function as well, proof of which can be found in [6].

For example, if we denote by ℓ an s.n. function determining the norm in ℓ^1 , then functions determining the norm in ℓ^p are exactly the p modifications $\ell^{(p)}$ of ℓ . More generally, by the basic properties of singular values, norm $\|\cdot\|_{\Phi^{(p)}}$ is given by the formula

$$\|X\|_{\Phi^{(p)}} \stackrel{\text{def}}{=} \left\| |X|^p \right\|_{\Phi}^{\frac{1}{p}} \quad \text{for all } X \in \mathcal{B}(\mathcal{H}) \text{ such that } |X|^p \in \mathcal{C}_\Phi(\mathcal{H}). \quad (1.1)$$

Specially, in the case $p = 2$ we get so called Q-norms, for which (1.1) is of the form

$$|||X|||_{(2)} = \left\| |X|^2 \right\|_{(2)}^{\frac{1}{2}} = |||X^*X|||_{(2)}^{\frac{1}{2}}, \quad X \in \mathcal{B}(\mathcal{H}) \quad \text{and} \quad X^*X \in \mathcal{C}_{|||\cdot|||}(\mathcal{H}).$$

We will also use notation $|||\cdot|||_{(p)}$ for p modification of the norm $|||\cdot|||$. Schatten tracial p norms, defined as $\|X\|_p \stackrel{\text{def}}{=} \sqrt[p]{\sum_{n=1}^\infty s_n^p(X)}$ for $1 \leq p < +\infty$, are exactly p modification of the trace norm $\|\cdot\|_1$.

Given $f, g \in \mathcal{H}$, we will use the notation $f \otimes g^*$ for one dimensional operators $f \otimes g^* : \mathcal{H} \rightarrow \mathcal{H} : h \mapsto \langle h, g \rangle f$ and their linear span is dense in each of $\mathcal{C}_p(\mathcal{H})$ for $1 \leq p \leq +\infty$. For a more complete account of the theory of norm ideals, the interested reader is referred to [3] and [9].

For an operator valued (o.v.) function $\mathcal{A} : \Omega \rightarrow \mathcal{B}(\mathcal{H}) : t \mapsto \mathcal{A}_t$, where $(\Omega, \mathfrak{M}, \mu)$ is a measure space (notation that will be used throughout the paper), we say to be weakly* measurable if $t \mapsto \langle \mathcal{A}_t f, g \rangle$ is measurable for all $f, g \in \mathcal{H}$ and weakly* integrable if $t \mapsto \langle \mathcal{A}_t f, g \rangle$ is in $L^1(\Omega, \mu)$ for all $f, g \in \mathcal{H}$. Actually, we can conclude a bit more, as the following lemma shows (see [8, Lemma 1.2])

Lemma 1.1 *Operator valued function $\mathcal{A} : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ is weakly* measurable (integrable) if and only if functions $t \mapsto \text{tr}(\mathcal{A}_t Y)$ are measurable (integrable) for every $Y \in \mathcal{C}_1(\mathcal{H})$.*

Before we proceed to the proof of lemma, we note that if functions $t \mapsto \text{tr}(\mathcal{A}_t Y)$ are integrable for every $Y \in \mathcal{C}_1(\mathcal{H})$, as $\mathcal{C}_1(\mathcal{H})^* \cong \mathcal{B}(\mathcal{H})$, there exists Gelfand, or weak* integral $\int_\Omega \mathcal{A} d\mu \in \mathcal{B}(\mathcal{H})$ such that

$$\text{tr} \left(\int_\Omega \mathcal{A}_t d\mu(t) Y \right) = \int_\Omega \text{tr}(\mathcal{A}_t Y) d\mu(t), \quad \text{for all } Y \in \mathcal{C}_1(\mathcal{H}). \quad (1.2)$$

Proof Proof we present can be found in [7].

Each one dimensional operator $f \otimes g^*$ is in $\mathcal{C}_1(\mathcal{H})$ and there holds $\text{tr}(\mathcal{A}_t(f \otimes g^*)) = \text{tr}(\mathcal{A}_t f \otimes g^*) = \langle \mathcal{A}_t f, g \rangle$, so measurability (integrability) of functions $t \rightarrow \text{tr}(\mathcal{A}_t Y)$ implies weak* measurability (integrability).

Conversely, if all scalar functions $t \mapsto \langle \mathcal{A}_t f, g \rangle$ are measurable, for $Y \in \mathcal{C}_1(\mathcal{H})$ with a Schmidt expansion given by $Y = \sum_{n=1}^{\infty} s_n(Y) f_n \otimes e_n^*$, we get

$$\text{tr}(\mathcal{A}_t Y) = \sum_{n=1}^{\infty} s_n(Y) \langle \mathcal{A}_t f_n, e_n \rangle. \quad (1.3)$$

By the absolute summability of the series in (1.3) (assured by $Y \in \mathcal{C}_1(\mathcal{H})$) we get that $\text{tr}(\mathcal{A}_t Y)$ is measurable as pointwise limit of measurable functions.

When all scalar functions $t \mapsto \langle \mathcal{A}_t f, g \rangle$ are additionally integrable, to prove integrability of $t \mapsto \text{tr}(\mathcal{A}_t Y)$ it will suffice to show that $\phi: \mathcal{H} \times \mathcal{H} \rightarrow L^1(\Omega, \mu)$ given by

$$\phi(f, g) = \langle \mathcal{A}_t f, g \rangle, \quad \text{for all } t \in \Omega \text{ and } f, g \in \mathcal{H} \quad (1.4)$$

defines a bounded sesquilinear mapping. Indeed in that case, by (1.3) function $t \mapsto \text{tr}(\mathcal{A}_t Y)$ will be a pointwise limit of $L^1(\Omega, \mu)$ absolutely summable series (guaranteed by the convergence of $\sum_{n=1}^{\infty} s_n(Y)$), and therefore integrable.

To prove that ϕ is closed in both variable, we start by proving it is closed in first one. Indeed, fix $g \in \mathcal{H}$ and define $\mathcal{J}_g: \mathcal{H} \rightarrow L^1(\Omega, \mu)$ by $\mathcal{J}_g f(t) = \langle \mathcal{A}_t f, g \rangle$. Clearly, $\mathcal{J}_g$ is linear, but also a closed mapping because $\|f_n - f\| + \|\mathcal{J}_g f_n - h\|_{L^1} \rightarrow 0$ for some $f, f_n \in \mathcal{H}, n = 1, 2, \dots$ and $h \in L^1(\Omega, \mu)$, first implies (with passing to a convergent subsequence if necessary) that $\lim_{n \rightarrow \infty} \mathcal{J}_g f_n(t) = h(t)$ for almost every $t \in \Omega$. For those $t \in \Omega$, as $\mathcal{A}_t$ is bounded operator, we also have $\langle \mathcal{A}_t f_n, g \rangle \rightarrow \langle \mathcal{A}_t f, g \rangle$ as $n \rightarrow \infty$. Therefore $\mathcal{J}_g f(t) = h(t)$ for almost every $t \in \Omega$, and by the closed graph Theorem, $\mathcal{J}_g$ is bounded, i.e. for some constant $C_g < +\infty$ holds

$$\|\langle \mathcal{A}_t f, g \rangle\|_{L^1} = \|\mathcal{J}_g f\| \leq C_g \|f\|, \quad \text{for all } f \in \mathcal{H}.$$

By analogy, family $\{\mathcal{I}_f\}_{f \in \mathcal{H}} \in \mathcal{B}(\mathcal{H}, L^1(\Omega, \mu))$ defined by $\mathcal{I}_f g(t) = \langle \mathcal{A}_t^* g, f \rangle$, for all $g \in \mathcal{H}$. Its subfamily $\{\mathcal{I}_f\}_{\|f\| \leq 1}$ is pointwise bounded as

$$\sup_{\|f\| \leq 1} \|\mathcal{I}_f g\|_{L^1} = \sup_{\|f\| \leq 1} \|\langle \mathcal{A}_t f, g \rangle\|_{L^1} = \sup_{\|f\| \leq 1} \|\mathcal{J}_g f\| \leq C_g < +\infty, \quad \text{for every } g \in \mathcal{H}.$$

By the uniform boundedness principle we conclude $M = \sup_{\|f\| \leq 1} \|\mathcal{I}_f\|_{\mathcal{B}(\mathcal{H}, L^1)} < +\infty$ and therefore

$$\sup_{\|f\|, \|g\| \leq 1} \|\langle \mathcal{A}_t f, g \rangle\|_{L^1} = \sup_{\|f\|, \|g\| \leq 1} \|\langle \mathcal{A}_t^* g, f \rangle\|_{L^1} \leq \sup_{\|f\| \leq 1} \|\mathcal{I}_f\|_{\mathcal{B}(\mathcal{H}, L^1)} = M,$$

as proclaimed. $\square$

Hence, for a weakly* integrable function $\mathcal{A}$ there exist Gel'fand integral $\int_{\Omega} \mathcal{A} d\mu \in \mathcal{B}(\mathcal{H})$, and specially

$$\left\langle \int_{\Omega} \mathcal{A} d\mu f, g \right\rangle = \int_{\Omega} \langle \mathcal{A}_t f, g \rangle d\mu(t),$$

for all $f, g \in \mathcal{H}$, and this is exactly the relation that entirely defines the operator $\int_{\Omega} \mathcal{A} d\mu$.

Before we proceed further, we note that if a function $\mathcal{A}$ takes values in some ideal of compact operators, we can define different weak* integral, if the mentioned ideal have a predual. Nevertheless, in this paper, we will only be interested in integrals defined by (1.2), i.e. when $\mathcal{A}$ is considered as $\mathcal{B}(\mathcal{H})$ valued function.

Every weakly* integrable function $\mathcal{A}$ defines a vector valued measure ν on $\mathcal{B}(\mathcal{H})$ by

$$\nu(E) = \int_E \mathcal{A}_t d\mu(t).$$

As function $t \mapsto \langle \mathcal{A}_t f, g \rangle$ belongs to $L^1(\Omega, \mu)$, for disjoint $E_n \in \mathfrak{M}$, $n \in \mathbb{N}$ we have $\int_{\bigsqcup_{n \in \mathbb{N}} E_n} \langle \mathcal{A}_t f, g \rangle d\mu(t) = \sum_{n=1}^{\infty} \int_{E_n} \langle \mathcal{A}_t f, g \rangle d\mu(t)$, hence the measure ν is weakly additive, but generally ν is only finitely additive. In the case

$$\nu\left(\bigsqcup_{n \in \mathbb{N}} E_n\right) = \sum_{n=1}^{\infty} \nu(E_n)$$

in the norm topology, we say that a measure ν is countably additive.

Also, if we denote by $B(\Omega, \mu)$ the space of simple functions with uniform norm, with vector measure ν we have defined operator of integration $T: B(\Omega, \mu) \rightarrow \mathcal{B}(\mathcal{H})$ by

$$Tf = T\left(\sum_{k=1}^n c_k \chi_{E_k}\right) = \sum_{k=1}^n c_k \nu(E_k).$$

Direct calculation shows that T is bounded by $\sup_{\|f\|=\|g\|=1} \int_{\Omega} |\langle \mathcal{A}_t f, g \rangle| d\mu(t)$ and therefore T also defines an operator on $L^{\infty}(\Omega, \mu)$, which we will also denote by T . We will also use notation $Tf = \int_{\Omega} f d\nu$. More details about mentioned results and vector measures in general, as well as different types of integration of vector valued functions can be found in [2].

An example of the very useful weak* integral is $\int_{\Omega} \mathcal{A}^* \mathcal{A} d\mu = \int_{\Omega} |\mathcal{A}|^2 d\mu$, with the associated quadratic form $\langle \int_{\Omega} \mathcal{A}^* \mathcal{A} d\mu f, f \rangle = \int_{\Omega} \|\mathcal{A}_t f\|^2 d\mu(t)$ for all $f \in \mathcal{H}$, provided by the finiteness of the last term in expression. Some constructions involving this example can be found in [5]. Motivated by this example, as well as results in [5], throughout this paper we are generally interested in functions $\mathcal{A}$ that consists of positive operators. Specially we are interested in results that provides that $\int_{\Omega} \mathcal{A} d\mu$ belongs to some ideal, as well as implications of that inclusion.

For the sake of completeness, we state the next technical result for positive operators. More general result for Schatten ideals can be found in [9, Theorem 2.16].

Lemma 1.2 *Let $A_n \in \mathcal{B}(\mathcal{H})$, for $n \in \mathbb{N}$, be sequence of positive operators that converges weakly to $A \in \mathcal{B}(\mathcal{H})$ and let there exist an operator $B \in \mathcal{C}_{|||\cdot|||}(\mathcal{H})$, where $\mathcal{C}_{|||\cdot|||}(\mathcal{H})$ is separable ideal, such that $A_n \leq B$, for every $n \in \mathbb{N}$. Then $|||A_n - A||| \rightarrow 0$, as $n \rightarrow \infty$.*

Proof Proof is essentially the same as the proof of [9, Theorem 2.16], so we provide only the essential steps.

First, by letting $n \rightarrow \infty$ in $\langle A_n f, f \rangle \leq \langle B f, f \rangle$ we get $A \leq B$ and moreover, from monotonicity properties of $|||\cdot|||$ we also get $A_n, A \in \mathcal{C}_{|||\cdot|||}(\mathcal{H})$, for all $n \in \mathbb{N}$.

Since $\mathcal{C}_{|||\cdot|||}(\mathcal{H})$ is separable, there exists a finite rank projection P such that for $Q = I - P$ we have that $|||QBQ||| < \varepsilon$. Moreover, from the $QA_nQ \leq QBQ$ and $QAQ \leq QBQ$ we also get $|||QA_nQ||| < \varepsilon$ and $|||QAQ||| < \varepsilon$. By the [1, Cor IV.2.6] we also have

$$\begin{aligned} |||A_nQ||| &= |||\sqrt{A_n}\sqrt{A_n}Q||| \leq |||\sqrt{A_n}|||_{(2)} |||\sqrt{A_n}Q|||_{(2)} \\ &= |||A_n|||^{1/2} |||QA_nQ|||^{1/2} \leq |||B|||^{1/2} \varepsilon^{1/2}. \end{aligned}$$

similarly, we also get $|||AQ||| \leq |||B|||^{1/2} \varepsilon^{1/2}$. Finally, as P is a finite rank projection, by the weak convergence of A_n we get $|||P(A_n - A)P||| \rightarrow 0$, as $n \rightarrow \infty$.

By the triangle inequality we get

$$|||A_n - A||| \leq |||P(A_n - A)P||| + |||Q(A_n - A)||| + |||P(A_n - A)Q|||,$$

which converges to 0, from the previous remarks. $\square$

We note that separability of $\mathcal{C}_{|||\cdot|||}(\mathcal{H})$ is necessary in the previous lemma. For example, for positive $B \in \mathcal{C}_{|||\cdot|||}(\mathcal{H}) \setminus \mathcal{C}_{|||\cdot|||}^{(0)}(\mathcal{H})$, its Schmidt expansion does not converge to B in the norm of the ideal.

Previous lemma enables us to prove countable additivity of certain operator valued functions, more precisely

Theorem 1.3 *Let $\mathcal{A} : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ be weakly* integrable function, that consists of positive operators, such that $\int_{\Omega} \mathcal{A} d\mu \in \mathcal{C}_{|||\cdot|||}(\mathcal{H})$, where $\mathcal{C}_{|||\cdot|||}(\mathcal{H})$ is a separable ideal. Then:*

- $\mathcal{C}_{|||\cdot|||}(\mathcal{H})$ -valued measure $\nu(E) = \int_E \mathcal{A} d\mu$ is countably additive.
- In the case of Schatten ideals $\mathcal{C}_p(\mathcal{H})$, for disjoint $E_n \in \mathfrak{M}$ we also have $\sum_{n=1}^{\infty} |||\nu(E_n)|||_p^p < +\infty$.

Proof As noted before, measure ν is weakly additive, i.e. for each $f, g \in \mathcal{H}$, and for disjoint $E_n \in \mathfrak{M}$, where $n \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty} \int_{E_n} \langle \mathcal{A}_t f, g \rangle d\mu(t) = \int_{\bigsqcup_{n \in \mathbb{N}} E_n} \langle \mathcal{A}_t f, g \rangle d\mu(t). \quad (1.5)$$

Since $\mathcal{A}_t$ are positive, for any $E \in \mathfrak{M}$ we have

$$\left\langle \int_E \mathcal{A}_t d\mu(t) f, f \right\rangle = \int_E \langle \mathcal{A}_t f, f \rangle d\mu(t) \geq 0 \quad (1.6)$$

hence operator $\int_E \mathcal{A} d\mu$ is positive as well. Similarly, for measurable $E \subset F$ we get $\int_E \mathcal{A} d\mu \leq \int_F \mathcal{A} d\mu$. Therefore, from (1.5) and (1.6), by the Lemma 1.2 we get that $\sum_{n=1}^{\infty} \int_{E_n} \mathcal{A} d\mu$ converges in the norm of the ideal $\mathcal{C}_{\|\cdot\|}(\mathcal{H})$.

To prove b), we start by applying the inequality [4, Lemma 2.1.c)] to the identity $\sum_{n=1}^N \int_{E_n} \mathcal{A} d\mu = \int_{\bigsqcup_{n=1}^N E_n} \mathcal{A} d\mu$, for the convex function $f(t) = t^p$, where $p \geq 1$ and we get

$$\left\| \sum_{n=1}^N \left(\int_{E_n} \mathcal{A}_t d\mu(t) \right)^p \right\| \leq \left\| \left(\int_{\bigsqcup_{n=1}^N E_n} \mathcal{A}_t d\mu(t) \right)^p \right\|, \quad (1.7)$$

for every u. i. norm $\|\cdot\|$, where $N \in \mathbb{N}$ is fixed. Furthermore, if we denote $A_N = \int_{\bigsqcup_{n=1}^N E_n} \mathcal{A} d\mu$ and $A = \int_{\bigsqcup_{n=1}^{\infty} E_n} \mathcal{A} d\mu$, from $A_N \leq A$ we get $s_n(A_N) \leq s_n(A)$, for every $n \in \mathbb{N}$. Hence, from the positivity of A_N and A , we further get $s_n(A_N^p) = s_n^p(A_N) \leq s_n^p(A) = s_n(A^p)$ and therefore, for every u. i. norm we have $\|A_N^p\| \leq \|A^p\|$. By applying latter to the (1.7) in the case of nuclear norm $\|\cdot\|_1$, we get

$$\left\| \sum_{n=1}^N \left(\int_{E_n} \mathcal{A}_t d\mu(t) \right)^p \right\|_1 \leq \left\| \left(\int_{\bigsqcup_{n=1}^{\infty} E_n} \mathcal{A}_t d\mu(t) \right)^p \right\|_1 = \left\| \int_{\bigsqcup_{n=1}^{\infty} E_n} \mathcal{A}_t d\mu(t) \right\|_p^p.$$

Again, from the positivity of operators $\int_{E_n} \mathcal{A}_t d\mu(t)$ we have

$$\left\| \sum_{n=1}^N \left(\int_{E_n} \mathcal{A}_t d\mu(t) \right)^p \right\|_1 = \sum_{n=1}^N \left\| \left(\int_{E_n} \mathcal{A}_t d\mu(t) \right)^p \right\|_1 = \sum_{n=1}^N \left\| \int_{E_n} \mathcal{A}_t d\mu(t) \right\|_p^p,$$

and we finally get

$$\sum_{n=1}^N \left\| \int_{E_n} \mathcal{A}_t d\mu(t) \right\|_p^p \leq \left\| \int_{\bigsqcup_{n=1}^{\infty} E_n} \mathcal{A}_t d\mu(t) \right\|_p^p.$$

Since N was arbitrary, by letting $N \rightarrow \infty$ we get the desired result. $\square$

2 Main results

We start with the connection of $\mathcal{A}_t \in \mathcal{C}_{\|\cdot\|}(\mathcal{H})$ and $\int_{\Omega} \mathcal{A} d\mu \in \mathcal{C}_{\|\cdot\|}(\mathcal{H})$. In the case of positive nuclear operators question is answered by the next lemma.

Lemma 2.1 Let $\mathcal{A} : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ be weakly* integrable family, such that $\mathcal{A}_t$ are positive operators for almost every $t \in \Omega$. Then $\int_{\Omega} \mathcal{A} d\mu \in \mathcal{C}_1(\mathcal{H})$ if and only if $\mathcal{A}_t \in \mathcal{C}_1(\mathcal{H})$ for almost every t and $\int_{\Omega} \|\mathcal{A}_t\|_1 d\mu(t) < +\infty$.

Proof Since $\int_{\Omega} \mathcal{A} d\mu$ is positive operator, by the [3, Lemma 8.1] in chapter III, it belongs to $\mathcal{C}_1(\mathcal{H})$ if and only if $\sum_{n=1}^{\infty} \langle \int_{\Omega} \mathcal{A} d\mu e_n, e_n \rangle < +\infty$, where $\{e_n\}_{n=1}^{\infty}$ is any orthonormal base of $\mathcal{H}$. By the definition of the Gelfand integral we get

$$\sum_{n=1}^{\infty} \left\langle \int_{\Omega} \mathcal{A}_t d\mu(t) e_n, e_n \right\rangle = \sum_{n=1}^{\infty} \int_{\Omega} \langle \mathcal{A}_t e_n, e_n \rangle d\mu(t). \quad (2.1)$$

Since $\mathcal{A}_t$ are positive operators and therefore $\langle \mathcal{A}_t e_n, e_n \rangle \geq 0$, for almost every $t \in \Omega$ and every $n \in \mathbb{N}$, by the use of Beppo Levi Theorem, (2.1) is further equal to

$$\sum_{n=1}^{\infty} \int_{\Omega} \langle \mathcal{A}_t e_n, e_n \rangle d\mu(t) = \int_{\Omega} \sum_{n=1}^{\infty} \langle \mathcal{A}_t e_n, e_n \rangle d\mu(t). \quad (2.2)$$

Moreover, from the positivity of $\mathcal{A}_t$ we also have $\|\mathcal{A}_t\|_1 = \sum_{n=1}^{\infty} \langle \mathcal{A}_t e_n, e_n \rangle$. Hence, $\int_{\Omega} \mathcal{A} d\mu$ belongs to $\mathcal{C}_1(\mathcal{H})$ if and only if the scalar function $t \mapsto \|\mathcal{A}_t\|_1$ is integrable, which concludes the proof of Lemma. $\square$

In the other ideals of compact operators, previous Lemma does not hold, as the following example shows.

Example 2.2 Let $f_{n,k} : [0, 1] \rightarrow \mathbb{C}$, for $n, k \in \mathbb{N}$ and $1 \leq k \leq n$, be defined as

$$f_{n,k}(t) = \begin{cases} 1, & \frac{k-1}{n} \leq t \leq \frac{k}{n} \\ 0, & \text{elsewhere} \end{cases}$$

and let $(g_n)_{n=1}^{\infty}$ be a sequence consisting of $f_{n,k}$ in some rearrangement. We note that $g_n(t)$ diverges for every $t \in [0, 1]$ while $\int_0^1 g_n(t) dt \rightarrow 0$, as $n \rightarrow \infty$.

Let us define an o. v. function $\mathcal{A} : [0, 1] \rightarrow \mathcal{B}(\mathcal{H})$ with $\mathcal{A}_t = \sum_{n=1}^{\infty} g_n(t) e_n \otimes e_n^*$, where $\{e_n\}_{n=1}^{\infty}$ is some basis of $\mathcal{H}$. First, $t \mapsto \mathcal{A}_t$ is weakly* measurable since for every $f \in \mathcal{H}$ we get

$$\langle \mathcal{A}_t f, f \rangle = \sum_{n=1}^{\infty} g_n(t) \langle f, e_n \rangle \langle e_n, f \rangle = \sum_{n=1}^{\infty} g_n(t) |\langle f, e_n \rangle|^2, \quad (2.3)$$

which is measurable as a series of measurable functions. Moreover, since $g_n(t) \leq 1$ for every $t \in [0, 1]$ and every $n \in \mathbb{N}$, righthand side of (2.3) can be further estimated by

$$\sum_{n=1}^{\infty} g_n(t) |\langle f, e_n \rangle|^2 \leq \sum_{n=1}^{\infty} |\langle f, e_n \rangle|^2 \leq \|f\|^2,$$

which is integrable function on $[0, 1]$.

Finally, to calculate the Gel'fand integral of $\mathcal{A}$, we have

$$\left\langle \int_0^1 \mathcal{A}_t dt f, f \right\rangle = \int_0^1 \sum_{n=1}^{\infty} g_n(t) |\langle f, e_n \rangle|^2 dt = \sum_{n=1}^{\infty} \int_0^1 g_n(t) dt |\langle f, e_n \rangle|^2,$$

or equivalently $\int_0^1 \mathcal{A} d\mu = \sum_{n=1}^{\infty} (\int_0^1 g_n(t) dt) e_n \otimes e_n^*$. Since $\int_0^1 g_n(t) dt \rightarrow 0$, as $n \rightarrow \infty$, the operator $\int_0^1 \mathcal{A} d\mu$ is compact. Moreover, for every fixed t , sequence $g_n(t)$ consists of infinite number of 0 and 1. Therefore, $\mathcal{A}_t$ is orthogonal projection for every t whose range is infinite dimensional. Hence, $\mathcal{A}_t$ is not compact operator for any $t \in [0, 1]$.

We note that the $\int_{\Omega} \mathcal{A} d\mu$ from previous example belongs to $\mathcal{C}_p(\mathcal{H})$ for all $2 < p < +\infty$, as it is a diagonal operator whose diagonal entries are some permutation of the sequence

$$\left(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots, \underbrace{\frac{1}{n}, \dots, \frac{1}{n}}_n, \dots\right). \quad (2.4)$$

Hence, singular values of the operator $\int_{\Omega} \mathcal{A} d\mu$ form exactly the sequence (2.4). Therefore

$$\begin{aligned} \left\| \int_{\Omega} \mathcal{A}_t d\mu(t) \right\|_p^p &= \sum_{n=1}^{\infty} s_n^p \left(\int_0^1 \mathcal{A}_t d\mu(t) \right) \\ &= \sum_{m=1}^{\infty} \underbrace{\left(\frac{1}{m} \right)^p + \dots + \left(\frac{1}{m} \right)^p}_m = \sum_{m=1}^{\infty} \frac{1}{m^{p-1}} < +\infty, \end{aligned}$$

for $p > 2$. Similar examples can be constructed for all other ideals non-isomorphic to $\mathcal{C}_1(\mathcal{H})$.

We also note that the previous example is expected, firstly as the integrals involved are not Bochner. Secondly, as some ideals lack the Radon - Nikodym property, so even for Bochner integrals we don't have connection between $\mathcal{A}_t \in \mathcal{C}_{|||\cdot|||}(\mathcal{H})$ and $\int_{\Omega} \mathcal{A} d\mu \in \mathcal{C}_{|||\cdot|||}(\mathcal{H})$.

For the case of o. v. functions that takes values in compact operators, next theorem provides sufficient conditions for operator $\int_{\Omega} \mathcal{A} d\mu$ to be compact. Also, if the operators in question are positive, necessary conditions are provided as well. More precisely

Theorem 2.3 *Let $\mathcal{A} : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ be weakly* integrable family, where (Ω, μ) is finite measure space and $\mathcal{A}_t \in \mathcal{C}_{\infty}(\mathcal{H})$ for almost all $t \in \Omega$. Then:*

- a) *If, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $E \in \mathfrak{M}$, $\mu(E) < \delta$ implies $\left\| \int_E \mathcal{A} d\mu \right\| < \varepsilon$, then the operator $\int_{\Omega} \mathcal{A} d\mu$ is compact*

b) If the operators $\mathcal{A}_t$ are also positive for almost every $t \in \Omega$ and the operator $\int_{\Omega} \mathcal{A} d\mu$ is compact, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $E \in \mathfrak{M}$, $\mu(E) < \delta$ implies $\|\int_E \mathcal{A} d\mu\| < \varepsilon$.

Proof To prove a), let $h_n \in \mathcal{H}$ be sequence that weakly converges to 0. Since $\mathcal{A}_t$ is compact for almost every $t \in \Omega$, sequence $\|\mathcal{A}_t h_n\|$ converges to 0, for almost every t . For $\varepsilon > 0$ fixed, let $\delta > 0$ be as in the conditions of the Theorem. By the Egorov's Theorem, there exists a measurable X_δ such that $\|\mathcal{A}_t h_n\|$ uniformly converges to 0 on X_δ and $\mu(\Omega \setminus X_\delta) < \delta$.

By the definition of Gelfand integral, for the unit vector $f \in \mathcal{H}$ and $E \in \mathfrak{M}$ we have the following inequalities

$$\begin{aligned} \left| \left\langle \int_E \mathcal{A}_t d\mu(t) h_n, f \right\rangle \right| &= \left| \int_E \langle \mathcal{A}_t h_n, f \rangle d\mu(t) \right| \\ &\leq \int_E |\langle \mathcal{A}_t h_n, f \rangle| d\mu(t) \leq \int_E \|\mathcal{A}_t h_n\| d\mu(t). \end{aligned} \quad (2.5)$$

Therefore, by taking the supremum on the lefthandside of (2.5) we get

$$\sup_{\|f\|=1} \left| \left\langle \int_E \mathcal{A}_t d\mu(t) h_n, f \right\rangle \right| = \left\| \int_E \mathcal{A}_t d\mu(t) h_n \right\| \leq \int_E \|\mathcal{A}_t h_n\| d\mu(t).$$

Finally we get

$$\begin{aligned} \left\| \int_{\Omega} \mathcal{A}_t d\mu(t) h_n \right\| &\leq \left\| \int_{X_\delta} \mathcal{A}_t d\mu(t) h_n \right\| + \left\| \int_{\Omega \setminus X_\delta} \mathcal{A}_t d\mu(t) h_n \right\| \\ &\leq \int_{X_\delta} \|\mathcal{A}_t h_n\| d\mu(t) + \left\| \int_{\Omega \setminus X_\delta} \mathcal{A}_t d\mu(t) \right\| \|h_n\|. \end{aligned} \quad (2.6)$$

Since the sequence $(h_n)_{n=1}^\infty$ is weakly convergent it is bounded as well, i.e. $\|h_n\| \leq M$ and since $\mu(\Omega \setminus X_\delta) < \delta$ we have $\|\int_{\Omega \setminus X_\delta} \mathcal{A} d\mu\| < \varepsilon$. Therefore, second summand in (2.6) is smaller than $M\varepsilon$. For the first summand, as $\|\mathcal{A}_t h_n\|$ uniformly converges to 0, there exists $N \in \mathbb{N}$, such that for $n > N$ we have $\|\mathcal{A}_t h_n\| < \varepsilon$. Hence we get

$$\int_{X_\delta} \|\mathcal{A}_t h_n\| d\mu(t) + \left\| \int_{\Omega \setminus X_\delta} \mathcal{A}_t d\mu(t) \right\| \|h_n\| \leq \int_{X_\delta} d\mu(t) \varepsilon + M\varepsilon \leq \mu(\Omega) \varepsilon + M\varepsilon,$$

which concludes the proof a).

To prove b), let $\int_{\Omega} \mathcal{A} d\mu$ be compact operator. First, if we observe $t \mapsto \mathcal{A}_t$ as a $\mathcal{C}_\infty(\mathcal{H})$ valued function, by the definition of weak* integral (in $\mathcal{B}(\mathcal{H})$), $t \mapsto \text{tr}(\mathcal{A}_t Y)$ is measurable for every $Y \in \mathcal{C}_1(\mathcal{H}) \cong \mathcal{C}_\infty(\mathcal{H})^*$, or equivalently, $t \mapsto \mathcal{A}_t$ is weakly measurable. Moreover, from the integrability of the functions $t \mapsto \text{tr}(\mathcal{A}_t Y)$, we also get that $\mathcal{A}$ is Dunford integrable. Now, as $\mathcal{A}_t$ are positive, the operator $\int_{\Omega} \mathcal{A} d\mu$ is positive as well and, as already mentioned in the proof of Theorem 1.3, for every $E \in \mathfrak{M}$ we have $\int_E \mathcal{A} d\mu \leq \int_{\Omega} \mathcal{A} d\mu$. Therefore, $\int_E \mathcal{A} d\mu$ is compact for every

$E \in \mathfrak{M}$ and hence $\mathcal{A}$ is Pettis integrable. Finally, by the [2, Theorem 5] on page 53 we get the stated result. $\square$

Remark 2.4 In the part b) of the previous Theorem, assumption $\mathcal{A}_t \geq 0$, for almost every $t \in \Omega$, gives us that $\int_{\Omega} \mathcal{A} d\mu \in \mathcal{C}_{\infty}(\mathcal{H})$ implies $\int_E \mathcal{A} d\mu \in \mathcal{C}_{\infty}(\mathcal{H})$ for every $E \in \mathfrak{M}$. Since this is the only part of the proof where positivity is necessary, we note that the Theorem is applicable for all o.v. functions $\mathcal{A}$ (not necessary positive) if we assume that $\int_E \mathcal{A} d\mu \in \mathcal{C}_{\infty}(\mathcal{H})$ for every $E \in \mathfrak{M}$, i.e. if $\mathcal{A}$ is Pettis integrable.

More precisely, previous Theorem actually states that $\int_E \mathcal{A} d\mu$ is compact for every $E \in \mathfrak{M}$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $E \in \mathfrak{M}$, $\mu(E) < \delta$ implies $\|\int_E \mathcal{A} d\mu\| < \varepsilon$.

We also provide some sufficient conditions for the operator $\int_{\Omega} \mathcal{A} d\mu$ to be compact, without the assumption $\mathcal{A}_t \in \mathcal{C}_{\infty}(\mathcal{H})$ for almost every $t \in \Omega$.

Theorem 2.5 Let $\mathcal{A}: \Omega \rightarrow \mathcal{B}(\mathcal{H})$ be weakly* integrable such that there exists an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of Hilbert space $\mathcal{H}$ and finite set $\mathcal{I} \subset \mathbb{Z}$ such that

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\langle \mathcal{A}_t e_n, e_{n+i} \rangle| d\mu(t) = 0 \quad \text{for every } i \in \mathcal{I} \quad (2.7)$$

and

$$\sum_{\substack{i, j \in \mathbb{N} \\ i-j \notin \mathcal{I}}} \left(\int_{\Omega} |\langle \mathcal{A}_t e_j, e_i \rangle| d\mu(t) \right)^2 < +\infty. \quad (2.8)$$

Then, the operator $\int_{\Omega} \mathcal{A} d\mu$ is compact.

Proof By the definition of the Gel'fand integral, matrix representation of bounded operator $\int_{\Omega} \mathcal{A} d\mu$ in orthonormal basis $\{e_n\}_{n \geq 1}$ of Hilbert space $\mathcal{H}$ is

$$\int_{\Omega} \mathcal{A}_t d\mu(t) = \left(\int_{\Omega} \langle \mathcal{A}_t e_j, e_i \rangle d\mu(t) \right)_{i, j \in \mathbb{N}}.$$

Hence, the operator $\int_{\Omega} \mathcal{A} d\mu$ is of the form $B + \sum_{k \in \mathcal{I}} A_k$ where

$$\langle B e_j, e_i \rangle = \begin{cases} \int_{\Omega} \langle \mathcal{A}_t e_j, e_i \rangle d\mu(t), & \text{if } i - j \notin \mathcal{I} \\ 0, & \text{if } i - j \in \mathcal{I}. \end{cases},$$

while A_k are single diagonal operators whose entries are

$$\int_{\Omega} \langle \mathcal{A}_t e_n, e_{n+k} \rangle d\mu(t) \quad \text{for } n \geq \max\{1, 1 - k\}.$$

From the condition (2.7) of the theorem, as entries on the diagonal of A_k converge to 0, we get that A_k is bounded and compact because

$$A_k = \sum_{n=\max\{1, 1-k\}}^{\infty} \left[\int_{\Omega} \langle \mathcal{A}_t e_n, e_{n+k} \rangle d\mu(t) \right] e_{n+k} \otimes e_n^* \quad \text{for } k \in \mathcal{I}$$

where previous series converge in norm of space $\mathcal{B}(\mathcal{H})$. Operator B is Hilbert-Schmidt by (2.8) and therefore operator $\int_{\Omega} \mathcal{A} d\mu$ is compact as the finite sum of compact operators. $\square$

We note that the operator from Example 2.2 is compact as a direct consequence of previous Theorem.

We also give some applications of previous results.

Theorem 2.6 *Let $\mathcal{A} : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ be weakly* integrable, such that operators $\mathcal{A}_t$ are positive and $\int_{\Omega} \mathcal{A} d\mu$ is compact. Then the operator $T : L^{\infty}(\Omega, \mu) \rightarrow \mathcal{B}(\mathcal{H})$ defined by*

$$Tf = \int_{\Omega} f(t) d\nu(t),$$

where $\nu(E) = \int_E \mathcal{A} d\mu$, is compact.

Proof As already mentioned in the proof of the Theorem 1.3, for every $E \in \mathfrak{M}$, we have $\nu(E) \leq \nu(\Omega)$. Therefore $\nu(\mathfrak{M}) = \{\nu(E) \mid E \in \mathfrak{M}\} \subset \{A \in \mathcal{B}(\mathcal{H}) \mid 0 \leq A \leq \nu(\Omega)\}$.

Next, we note that the set $K = \{A \in \mathcal{B}(\mathcal{H}) \mid 0 \leq A \leq B\}$ is compact in uniform topology for every positive compact operator B . First, as K is bounded, by Alaoglu Theorem it is relatively compact in weak* topology and also in the weak topology, as weak* and weak topology coincide on the bounded sets. Moreover, since $\mathcal{H}$ is separable, weak topology is metrizable and therefore any sequence $(A_n)_{n=1}^{\infty}$ in K has weakly convergent subsequence A_{n_k} . Finally by the Lemma 1.2 A_{n_k} converges in norm as well, hence K is compact.

As in the proof of [2, Theorem 8.b)] on page 55, every simple function $g = \sum_{k=1}^n \alpha_k \chi_{E_k}$, where $0 \leq \alpha_k \leq \alpha_{k+1} \leq 1$, can be written in the form $g = \alpha_1 \sum_{k=1}^n \chi_{E_k} + \sum_{k=2}^n (\alpha_k - \alpha_{k-1}) \sum_{j=k}^n \chi_{E_k}$. Hence, set $T(\{\sum_{k=1}^n \alpha_k \chi_{E_k} \mid 0 \leq \alpha_k \leq 1, E_k \in \mathfrak{M}\})$ is contained in the convex hull of the compact set K and therefore is relatively compact set. Hence, operator T is compact. $\square$

Acknowledgements Authors are grateful to professor Danko Jocić for providing them a proof of lemma 1.1 in private communication.

Author Contributions All authors have equally contributed to writing and preparing the manuscript text. Author S. M. mainly contributed to formulation and proof of Theorem 1.2. while authors M. M. and M. K. contributed mainly to formulation and proof of Theorem 2.3. All authors reviewed the results and approved the final version of the manuscript.

Funding All authors have been funded by MPNTR grant, No. 174017, Serbia.

Declarations

Conflict of interest No potential conflict of interests was reported by the authors.

References

1. Bhatia, R.: Matrix Analysis. Springer-Verlag, New York (1997)
2. Diestel, J., Uhl, J.J.: Vector Measures, Amer. Math. Soc., Mat. Surveys and Monographs, Vol.15, (1977)
3. Gohberg, I.C., Krein, M.G.: Introduction to the Theory of Linear Non-selfadjoint Operators, Transl. Math. Monographs, Vol. 18, Amer. Math. Soc. Providence, R.I. (1969)
4. Hirzallah, O., Kittaneh, F.: Clarkson Inequalities for n-Tuples of Operators. *Integ. Equ. Oper. Theory.* **60**, 369–379 (2008)
5. Jocić, D.R.: Cauchy-Schwarz norm inequalities for weak*-integrals of operator valued functions. *J. Funct. Anal.* **218**, 318–346 (2005)
6. Jocić, D.R.: Multipliers of elementary operators and comparison of row and column space Schatten p norms. *Linear Alg. Appl.* **431**, 2062–2070 (2009)
7. Jocić, D.R.: *Unpublished paper*
8. Jocić, D.R., Đ. Krtinić, Moslehian, M.S.: *Landau and Grüss type inequalities for inner product type integral transformers in norm ideals*, *Math. Inequalities and Appl.*, 1 (2013), 109–125
9. Simon, B.: Trace Ideals and Their Applications, Amer. Math. Soc., Mat. Surveys and Monographs, Vol.120, (2005)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.