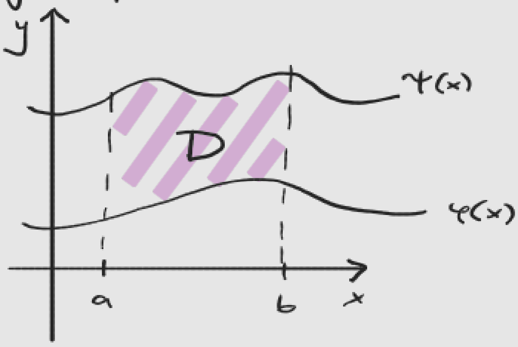
* Двоструки интеграл

(1) $D=\left\{(x,y)\in\mathbb{R}^2 \;\middle|\; a\le x\le b,\ \varphi(x)\le y\le \psi(x)\right\}$

y је огр. неким функцијама од x

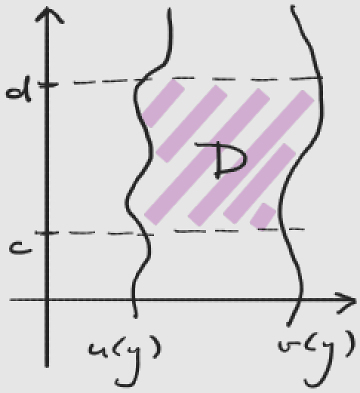
f непр. на D



Фубини $\Longrightarrow$ $\displaystyle\iint_D f(x,y)\,dx\,dy=\int_a^b\left(\int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy\right)dx$

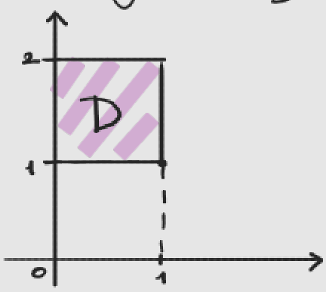
* $f\equiv 1 \rightsquigarrow$ интеграл је површина од D

(2) $D=\left\{(x,y)\in\mathbb{R}^2 \;\middle|\; c\le y\le d,\ u(y)\le x\le v(y)\right\}$



Фубини $\Longrightarrow$ $\displaystyle\iint_D f(x,y)\,dx\,dy=\int_c^d\left(\int_{u(y)}^{v(y)} f(x,y)\,dx\right)dy$

1. Израчунати $\iint_D (x+y)\,dx\,dy$, при чему је $D=[0,1]\times[1,2]$.



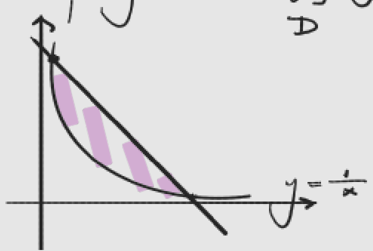
$$\iint_D (x+y)\,dx\,dy=\int_0^1\left(\int_1^2 (x+y)\,dy\right)dx=\int_0^1\left(x\cdot y\Big|_1^2+\frac{y^2}{2}\Big|_1^2\right)dx$$

$$=\int_0^1 x+\frac{3}{2}\,dx$$

$$=\frac{x^2}{2}\Big|_0^1+\frac{3}{2}x\Big|_0^1=\frac12+\frac32=2$$

□

2. Израчунати $\iint_D xy\,dx\,dy$, где је D скуп ограничен са $\alpha: x+y=\frac52$, $\beta: xy=1$



пресечне тачке:

$$x+\frac{1}{x}=\frac{5}{2}\quad /\cdot x$$

$$x^2-\frac52x+1=0$$

$$x_{1,2}=\frac{\frac52\pm\sqrt{\frac{25}{4}-4}}{2}=\frac{\frac52\pm\frac32}{2}$$

$\rightsquigarrow$ $x_1 = \frac{1}{2}$, $x_2 = 2$

$\downarrow \qquad\qquad \downarrow$

$y_1 = 2 \qquad y_2 = \frac{1}{2}$

$$\iint_D xy\,dy = \int_{\frac12}^{2}\int_{\frac1x}^{\frac52 - x} xy\,dy\,dx = \int_{\frac12}^{2} x\cdot\frac{y^2}{2}\Big|_{\frac1x}^{\frac52-x} dx = \frac12\int_{\frac12}^{2} x\left(\frac{25}{4} - 5x + x^2 - \frac{1}{x^2}\right)dx$$

$$= \frac12\left(\frac{25}{4}\cdot\frac{x^2}{2}\Big|_{\frac12}^{2} - 5\frac{x^3}{3}\Big|_{\frac12}^{2} + \frac{x^4}{4}\Big|_{\frac12}^{2} - \ln x\Big|_{\frac12}^{2}\right)$$

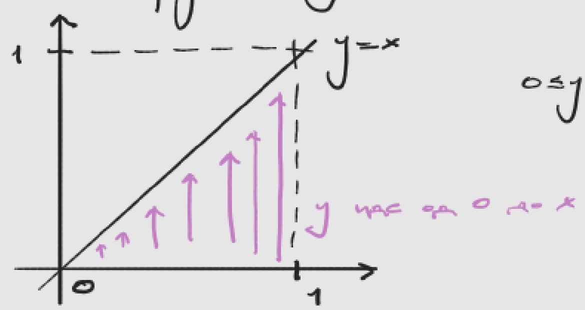
$$= \frac{165}{128} - \ln 2$$

□

3. Променити редослед интеграције.

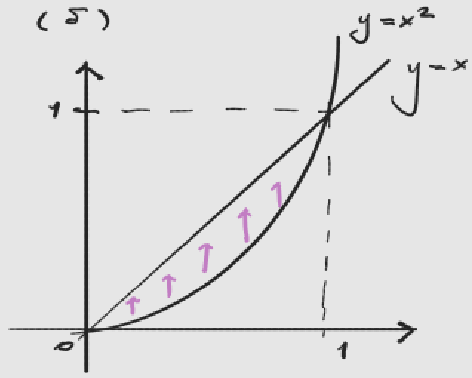
(а) $\int_0^1\left(\int_0^x f(x,y)\,dy\right)dx$ (б) $\int_0^1\left(\int_{x^2}^{x} f(x,y)\,dy\right)dx$

(а) Скицирајмо скуп:



$0 \le y \le x \le 1$

(б)



$$\rightsquigarrow \int_0^1\left(\int_0^x f(x,y)\,dy\right)dx = \int_0^1\left(\int_y^1 f(x,y)\,dx\right)dy$$

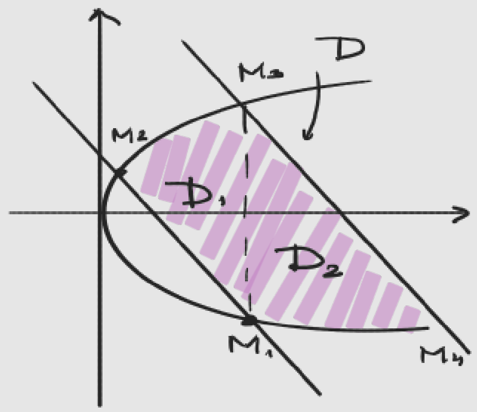
$0 \le \underbrace{x^2 \le y}_{x\le\sqrt{y}} \le x \le 1$

$x \le \sqrt{y}$ $\rightsquigarrow$ $y \le x \le \sqrt{y}$

$$\rightsquigarrow \int_0^1\left(\int_y^{\sqrt{y}} f(x,y)\,dx\right)dy$$

□

4. Израчунати $\iint_D (x+y)\,dx\,dy$, где је D ограничен са $y^2 = 2x$, $x+y=4$, $x+y=12$



пресечне тачке:

$\left.\begin{array}{l} x+y=4 \\ y^2 = 2x \end{array}\right\} \rightsquigarrow y^2 = 2(4-y)$

$y^2 + 2y - 8 = 0$

$y_{1,2} = \frac{-2 \pm \sqrt{4+32}}{2}$

$y_1 = -4 \qquad y_2 = 2 \rightsquigarrow M_1(8,-4)$, $M_2(2,2)$

И слично за $x+y=12$, $y^2=2x \rightsquigarrow M_3(8,4)$, $M_4(18,-6)$

Поделимо D на 2 области: D_1 и D_2 (интеграл је адитиван!)

$$\left(\iint_D = \iint_{D_1} + \iint_{D_2}\right)$$

$$\iint_D (x+y)\,dxdy = \iint_{D_1}(x+y)\,dxdy + \iint_{D_2}(x+y)\,dxdy$$

$$= \int_2^8 \Big(\int_{4-x}^{\sqrt{2x}} (x+y)\,dy \Big)\,dx + \int_8^{18} \Big(\int_{-\sqrt{2x}}^{12-x} (x+y)\,dy \Big)\,dx = \ldots$$

позитиван део параболе

негативан део параболе

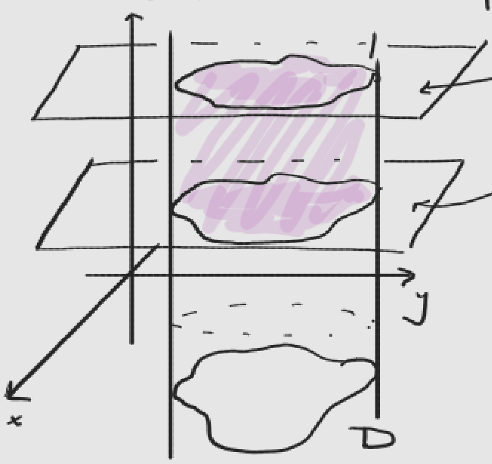
(завршити за вежбу)

$\square$

* Троструки интеграл

$D \subseteq \mathbb{R}^2$ – ограничен

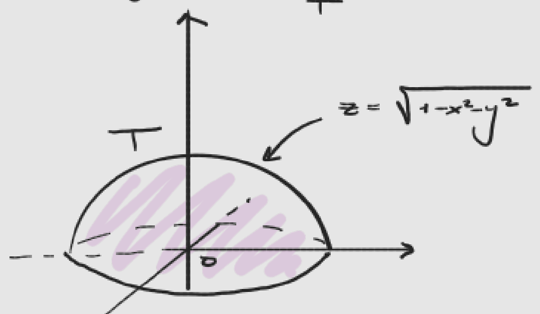
$$T_z = \Big\{ (x,y,z) \in \mathbb{R}^3 \;\Big|\; \overbrace{\varphi(x,y) \le z \le \psi(x,y)}^{z\text{ – координата између 2 површи}},\ (x,y) \in D \Big\}$$

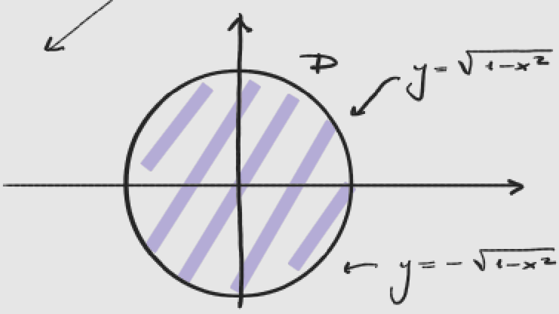


Фубини: $$\iiint_T f(x,y,z)\,dxdydz = \iint_D \Big(\int_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z)\,dz \Big)\,dxdy$$

* $f=1 \rightsquigarrow$ интеграл је запремина тела T

(5.) Израчунати $\iiint_T z\,dxdydz$, при чему је $T = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2 \le 1 \ \wedge\ z \ge 0\}$





$$I = \iint_D \Big(\int_0^{\sqrt{1-x^2-y^2}} z\,dz \Big)\,dxdy$$

$$= \iint_D \frac{1-x^2-y^2}{2}\,dx\,dy$$

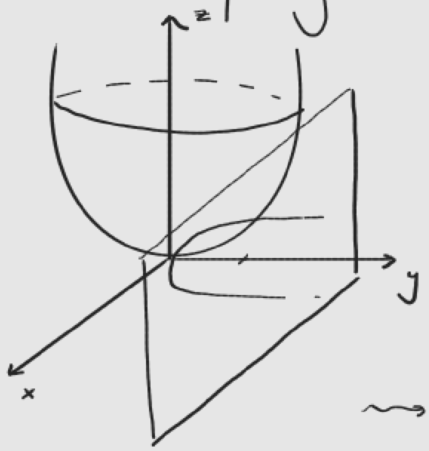
$$= \int_{-1}^{1} \Big(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1-x^2-y^2}{2}\,dy \Big)\,dx$$

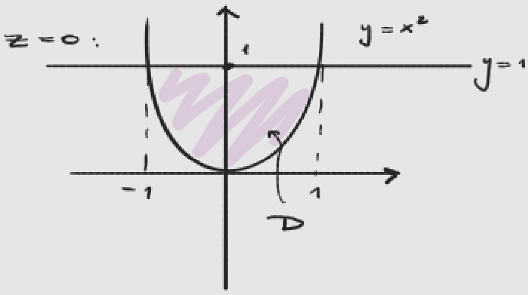
$$= \ldots = \{\text{СМЕНА: } x=\sin t\} = \ldots = \frac{\pi}{4}$$

$+ \cos^2 t = \frac{1+\cos 2t}{2}$

$\square$

6. Наћи запремину тела ограниченог са $z=x^2+y^2$, $y=x^2$, $y=1$ и $z=0$



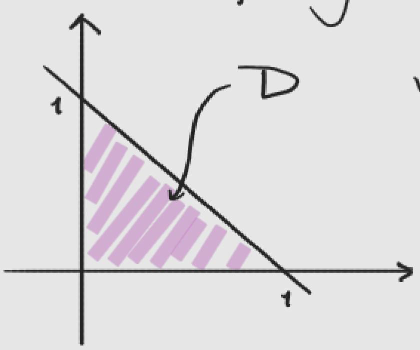


$$\rightsquigarrow \iiint_T dx\,dy\,dz = \iint_D \left(\int_0^{x^2+y^2} dz \right) dx\,dy = \iint_D (x^2+y^2)\,dx\,dy = \int_{-1}^{1} \left(\int_{x^2}^{1} (x^2+y^2)\,dy \right) dx$$

$$= \dots = 2\left(\frac{1}{3} - \frac{1}{5} + \frac{1}{3} - \frac{1}{21}\right)$$

$\square$

7. Наћи запремину тела $T = \{(x,y,z) \in \mathbb{R}^3 \mid x,y,z \geq 0,\ x+y \leq 1,\ z \leq x\sqrt{x} + y\sqrt{y}\}$



$$V(T) = \iiint_T dx\,dy\,dz = \iint_D \left(\int_0^{x\sqrt{x}+y\sqrt{y}} dz \right) dx\,dy$$

$$= \iint_D (x\sqrt{x} + y\sqrt{y})\,dx\,dy$$

$$= \int_0^1 \left(\int_0^{1-x} (x\sqrt{x} + \underbrace{y\sqrt{y}}_{y^{\frac{3}{2}}})\,dy \right) dx$$

$$= \int_0^1 \left(x^{\frac{3}{2}} \cdot (1-x) + \frac{2}{5} y^{\frac{5}{2}} \Big|_0^{1-x} \right) dx = \int_0^1 \left(x^{\frac{3}{2}} - x^{\frac{5}{2}} + \frac{2}{5}(1-x)^{\frac{5}{2}} \right) dx$$

$$= \dots = \frac{8}{35}$$

$\square$

Т. (о смени променљиве) $D_t, D_x \subseteq \mathbb{R}^n$ − ограничени, отворени, повезани,

$\varphi : D_t \longrightarrow D_x$ дифеоморфизам (бијекција класе C^1 т.д. је и $f^{-1} \in C^1$)

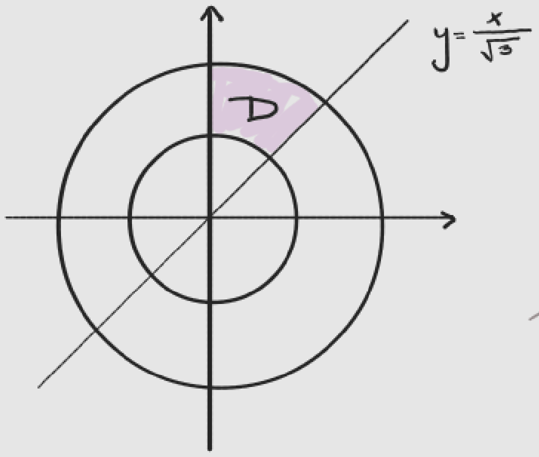
$f : D_x \longrightarrow \mathbb{R}$, $\operatorname{supp} f = \{x \in D_x \mid f(x) \neq 0\}$ компактан, $f \in \mathcal{R}(D_x)$

носач

Риман − интеграбилна

$$\boxed{\int_{D_x} f(x_1, x_2, \dots, x_n)\,dx_1 \dots dx_n = \int_{D_t} f \circ \varphi(t_1, \dots, t_n)\,\left|\det d\varphi(t_1, \dots, t_n)\right|\,dt_1 \dots dt_n}$$

⑧ Израчунати интеграл $\iint_D e^{x^2+y^2}\,dx\,dy$, $a,b>0$, $b>a$, $D=\left\{(x,y)\in\mathbb{R}^2 \;\middle|\; a^2\le x^2+y^2\le b^2,\ x\ge 0,\ y\ge \frac{x}{\sqrt3}\right\}$



смена: поларне координате

$x=r\cos\theta \qquad a\le r\le b$

$y=r\sin\theta \qquad \frac{\pi}{6}\le\theta\le\frac{\pi}{2}$

$$\varphi(r,\theta)=(\underbrace{r\cos\theta}_{\varphi_1},\underbrace{r\sin\theta}_{\varphi_2})$$

$$d\varphi(r,\theta)=\begin{vmatrix}\frac{\partial\varphi_1}{\partial r} & \frac{\partial\varphi_1}{\partial\theta}\\ \frac{\partial\varphi_2}{\partial r} & \frac{\partial\varphi_2}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}$$

$$|\det d\varphi(r,\theta)|=|r\cos^2\theta+r\sin^2\theta|=r$$

Јакобијан за поларне координате!

$$\rightsquigarrow \iint_D e^{x^2+y^2}\,dx\,dy=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_a^b e^{r^2}\cdot r\,dr\,d\theta=\left(\frac{\pi}{2}-\frac{\pi}{6}\right)\cdot\int_a^b e^{r^2}\cdot r\,dr=\left\{\begin{array}{l}s=r^2\\ ds=2r\,dr\end{array}\right\}$$

$$=\frac{\pi}{6}\cdot\int_{a^2}^{b^2}e^s\,ds=\frac{\pi}{6}\left(e^{b^2}-e^{a^2}\right)$$

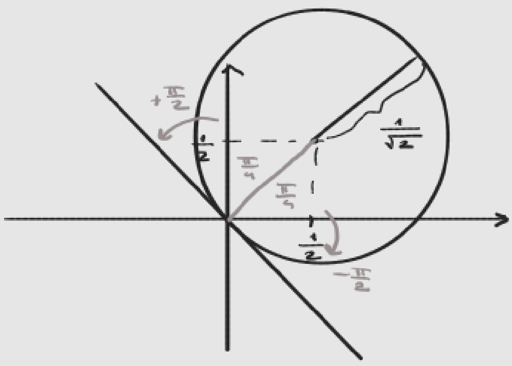
□

⑨ Израчунати интеграл $\iint_D (x+y)\,dx\,dy$, $D=\left\{(x,y)\in\mathbb{R}^2 \;\middle|\; x^2+y^2\le x+y\right\}$

$$x^2-x+y^2-y\le 0$$

$$x^2-2\cdot\tfrac12\cdot x+\tfrac14-\tfrac14+y^2-2\cdot\tfrac12 y+\tfrac14-\tfrac14\le 0$$

$$\left(x-\tfrac12\right)^2+\left(y-\tfrac12\right)^2\le\tfrac12=\left(\tfrac{1}{\sqrt2}\right)^2$$



I начин: $x=r\cos\theta \qquad \frac{\pi}{4}-\frac{\pi}{2}\le\theta\le\frac{\pi}{4}+\frac{\pi}{2}$

$y=r\sin\theta \qquad \rightsquigarrow -\frac{\pi}{4}\le\theta\le\frac{3\pi}{4}$

$0\le r\le \underset{\sqcup}{?}$ није јасна граница за r

Мењамо у једначину кружнице: $x^2+y^2=x+y$

$r^2=r(\cos\theta+\sin\theta)\rightsquigarrow r=\cos\theta+\sin\theta$

$\Longrightarrow 0\le r\le\cos\theta+\sin\theta$

$$\rightsquigarrow I = \iint_D (x+y)\,dx\,dy = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^{\cos\theta+\sin\theta} r\cdot(\cos\theta+\sin\theta)\cdot \underbrace{r}_{\substack{1 \\ \text{Јакобијан!}}}\,dr\,d\theta = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} (\cos\theta+\sin\theta)\cdot \frac{r^3}{3}\Big|_0^{\cos\theta+\sin\theta} d\theta = \dots$$

II начин: ПОМЕРЕНЕ ПОЛАРНЕ КООРДИНАТЕ

$$x-\tfrac{1}{2} = r\cos\theta \rightsquigarrow x = \tfrac{1}{2} + r\cos\theta \qquad J = r \text{ – Јакобијан}$$

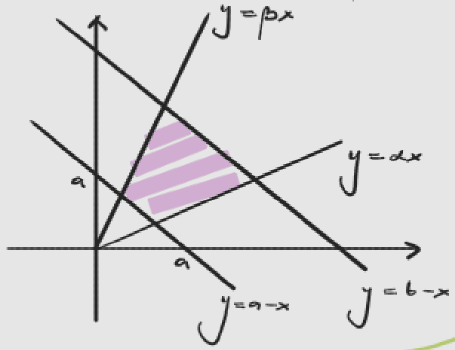
$$y-\tfrac{1}{2} = r\sin\theta \rightsquigarrow y = \tfrac{1}{2} + r\sin\theta$$

$$\left.\begin{array}{l} -\pi \le \theta \le \pi \\ 0 \le r \le \frac{1}{\sqrt{2}} \end{array}\right\} \text{ једноставније границе!}$$

$$I = \int_{-\pi}^{\pi} \int_0^{\frac{1}{\sqrt{2}}} \left(1 + r(\cos\theta+\sin\theta)\right)\cdot r\,dr\,d\theta = \int_{-\pi}^{\pi} \left(\frac{1}{4} + (\cos\theta+\sin\theta)\cdot\frac{1}{6\sqrt{2}}\right) d\theta = \dots = \frac{\pi}{2}$$

□

10. Израчунати интеграл $\iint_D \frac{dx\,dy}{xy}$, при чему је $D = \left\{(x,y)\in\mathbb{R}^2 \,\middle|\, a \le x+y \le b,\ \alpha x \le y \le \beta x\right\}$, $0<a<b$, $0<\alpha<\beta$



$$\underbrace{a \le x+y \le b}_{u}, \qquad \alpha \le \underbrace{\frac{y}{x}}_{v} \le b$$

$$\text{смена: } \begin{cases} u = x+y, & a \le u \le b \\ v = \frac{y}{x}, & \alpha \le v \le \beta \end{cases}$$

Треба да изразимо старе координате преко нових да бисмо израчунали Јакобијан.

$$y = vx$$

$$u = x+y = x(v+1) \Rightarrow \boxed{x = \frac{u}{1+v}}$$

$$\Rightarrow \boxed{y = \frac{uv}{1+v}}$$

$$(1+v)^2 = \left(\frac{x+y}{x}\right)^2$$

$$\frac{x^2}{(x+y)^2}\cdot\cancel{(x+y)}$$

$$J = \left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right| = \left|\begin{vmatrix} \frac{1}{1+v} & \frac{-u}{(1+v)^2} \\ \frac{v}{1+v} & \frac{u(1+v)-uv}{(1+v)^2} \end{vmatrix}\right| = \left|\frac{u}{(1+v)^3} + \frac{uv}{(1+v)^3}\right| = \frac{u}{(1+v)^2}$$

$$\rightsquigarrow \iint_D \frac{dx\,dy}{xy} = \int_a^b \left(\int_\alpha^\beta \frac{\cancel{(1+v)^2}}{u^{\cancel{2}} v}\cdot\frac{\cancel{u}}{\cancel{(1+v)^2}}\,dv\right) du = \int_a^b \frac{1}{u}\,du \int_\alpha^\beta \frac{1}{v}\,dv = \ln\frac{b}{a}\cdot\ln\frac{\beta}{\alpha}$$

II начин за Јакобијан: (ако је компликовано изразити x и y преко u и v)

$$J = \frac{D(x,y)}{D(u,v)} = \frac{1}{\frac{D(u,v)}{D(x,y)}}$$