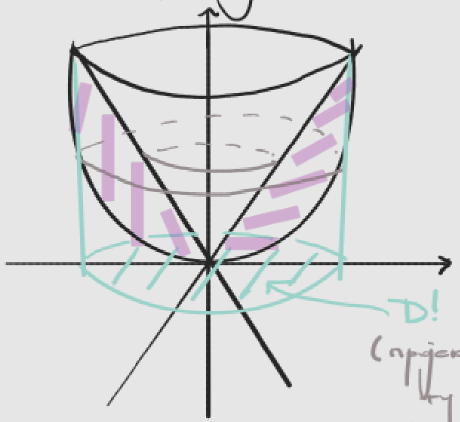
$$\frac{D(u,v)}{D(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{1}{x} + \frac{y}{x^2} = \frac{x+y}{x^2}$$

$$\rightsquigarrow J = \frac{x^2}{x+y} = \frac{x^2}{(x+y)^2}\cdot(x+y) = \underbrace{\left(\frac{x}{x+y}\right)^2}_{\left(\frac{1}{v+1}\right)^2}\cdot\underbrace{(x+y)}_{u} = \frac{u}{(1+v)^2}$$

∎

11. Наћи запремину тела T које је ограничено површима $z=\sqrt{x^2+y^2}$ и $az = x^2+y^2$, $a>0$.



(пројекција на раван $z=0$, ту се налазе x и y)

$z^2 = x^2+y^2$ – конус – овде само горњи део пошто је $z \geq 0$

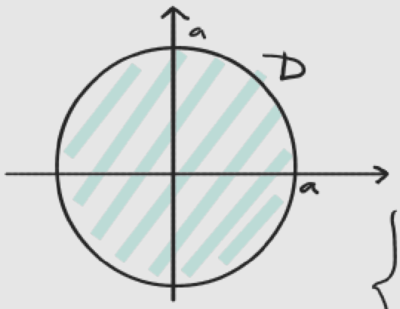
$az = x^2+y^2$ – параболоид

пресечне тачке: $\left.\begin{array}{l} z = \sqrt{x^2+y^2} \\ z = \frac{x^2+y^2}{a} \end{array}\right\} \sqrt{x^2+y^2} = \frac{x^2+y^2}{a}$

$\Longrightarrow \sqrt{x^2+y^2} = a$

$\rightsquigarrow x^2+y^2 = a^2 \rightsquigarrow$ круг. полупр. a

$\Longrightarrow z = a$



$$\iiint_T dx\,dy\,dz = \iint_D \int_{\frac{x^2+y^2}{a}\,\rightsquigarrow\,\text{до параболоида}}^{\sqrt{x^2+y^2}\,\rightsquigarrow\,\text{до конуса}} dz\; dx\,dy = \iint_D \left(\sqrt{x^2+y^2} - \frac{x^2+y^2}{a}\right)dx\,dy$$

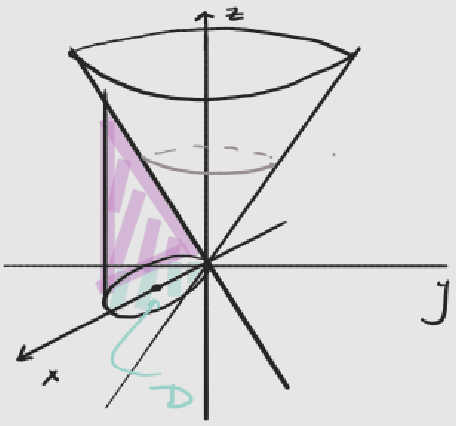
$$\left\{\begin{array}{ll} \text{поларне координате:} & \\ x = r\cos\theta, & 0 \leq r \leq a \\ y = r\sin\theta, & -\pi \leq \theta \leq \pi \end{array}\right\} = \int_{-\pi}^{\pi}\int_0^a \left(r - \frac{r^2}{a}\right)\cdot \overset{J}{r}\, dr\, d\theta$$

$$= 2\pi\cdot\left(\frac{a^3}{3} - \frac{a^3}{4}\right)$$

$$= \frac{a^3\pi}{6}$$

∎

12. Наћи запремину тела T ако је $T = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \geq z^2,\ x^2+y^2 \leq 2x,\ z \geq 0\}$



$x^2+y^2 = z^2$ – конус

$x^2+y^2 \geq z^2$ – спољашност конуса!

$x^2+y^2 = 2x$

$\Updownarrow$

$x^2 - 2x + 1 - 1 + y^2 = 0$

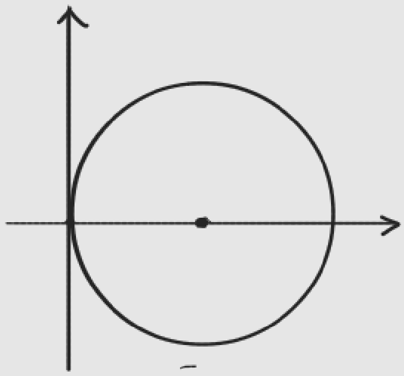
$(x-1)^2 + y^2 = 1 \rightsquigarrow$ круг са центром $(1,0)$

+ нема z коорд $\longrightarrow$ цилиндар!

$$V(T)=\iint\limits_D \left(\int_0^{\sqrt{x^2+y^2}} dz\right) dx\,dy = \iint\limits_D \sqrt{x^2+y^2}\,dx\,dy$$

I начин: Померене поларне – тешко...

II начин: стандардне поларне:



$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$x = r\cos\theta$

$y = r\sin\theta$

$0 \le r \le ?$

$x^2+y^2=2x \rightsquigarrow r^2 = 2r\cos\theta$

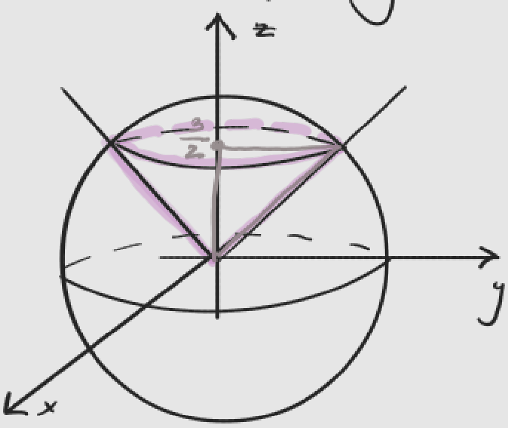
$\Rightarrow r = 2\cos\theta$

$\rightsquigarrow 0 \le r \le 2\cos\theta$, $J = r$

$$\iint\limits_D \sqrt{x^2+y^2}\,dx\,dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^{2\cos\theta} r\cdot r\,dr\,d\theta\right) = \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta\,d\theta = \dots = \frac{32}{9}$$

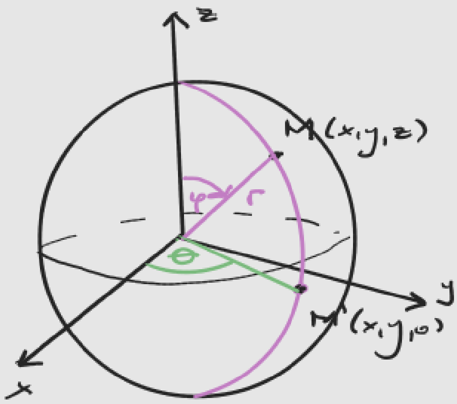
$\square$

13. Наћи запремину тела $T = \left\{ (x,y,z)\in\mathbb{R}^3 \;\middle|\; x^2+y^2+z^2 \le 9,\ z^2 \ge \frac{x^2+y^2}{3},\ z\ge 0 \right\}$



$3z^2 \ge x^2+y^2 \rightsquigarrow$ унутрашњост конуса

Користимо сферне координате:



$r \ge 0$

$-\pi \le \theta \le \pi$

$0 \le \varphi \le \pi$

$$\boxed{\begin{aligned} x &= r\sin\varphi\cos\theta \\ y &= r\sin\varphi\sin\theta \\ z &= r\cos\varphi \\ J &= r^2\sin\varphi \end{aligned}}$$

Код нас: $x = r\sin\varphi\cos\theta$ $\qquad 0 \le r \le 3$

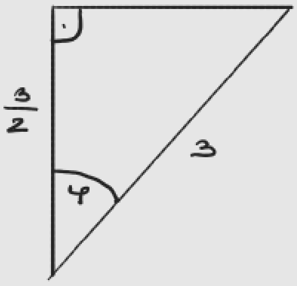
$y = r\sin\varphi\sin\theta$ $\qquad -\pi \le \theta \le \pi$

$z = r\cos\theta$ $\qquad 0 \le \varphi \le$ до конуса : треба нам пресек сфере и конуса:

$J = r^2\sin\varphi$

$$\left.\begin{aligned} x^2+y^2+z^2 &= 9 \\ 3z^2 &= x^2+y^2 \end{aligned}\right\} \Rightarrow 4z^2 = 9 \Rightarrow z = \pm\frac{3}{2}, \text{ али } z \ge 0$$

$$\Rightarrow z = \frac{3}{2}$$



$$\rightsquigarrow \cos\varphi = \frac{\frac{3}{2}}{3} = \frac{1}{2} \Rightarrow \varphi = \frac{\pi}{3}$$

$$\rightsquigarrow 0 \le \varphi \le \frac{\pi}{3}$$

$$V(T) = \iiint_T dx\,dy\,dz = \int_{-\pi}^{\pi}\int_0^{\frac{\pi}{3}}\int_0^3 r^2\sin\varphi\,dr\,d\varphi\,d\theta = \left(\int_{-\pi}^{\pi} d\theta\right)\cdot\left(\int_0^{\frac{\pi}{3}}\sin\varphi\,d\varphi\right)\cdot\left(\int_0^3 r^2\,dr\right) = 9\pi$$

$\square$

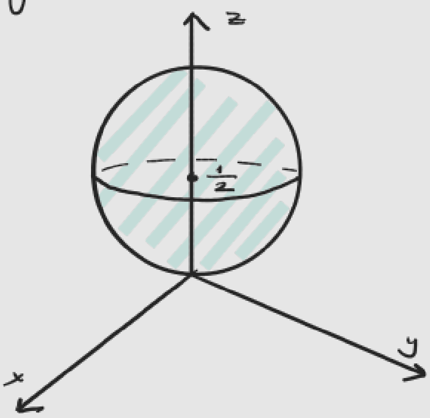
14. Израчунати $\iiint_T \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$, при чему је T ограничено са $x^2+y^2+z^2 = z$

$x^2+y^2+z^2 = z$

$x^2+y^2+z^2 - 2\cdot\frac{1}{2}z + \frac{1}{4} = \frac{1}{4}$

$x^2+y^2+\left(z-\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$

$\hookrightarrow$ сфера са центром у $(0,0,\frac{1}{2})$ полупречника $\frac{1}{2}$



I начин: померене сферне координате

$x = r\sin\varphi\cos\theta$ $\qquad 0 \le r \le \frac{1}{2}$ $\qquad I = \ldots$ (компликованији интеграл)

$y = r\sin\varphi\sin\theta$ $\qquad -\pi \le \theta \le \pi$

$z = \frac{1}{2} + r\cos\varphi$ $\qquad 0 \le \varphi \le \pi$

$J = r^2\sin\varphi$

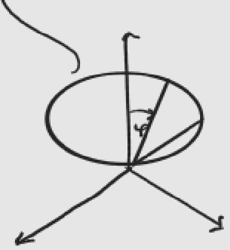
II начин: стандардне сферне координате

$x = r\sin\varphi\cos\theta$ $\qquad -\pi \le \theta \le \pi$ $\qquad x^2+y^2+z^2 = z$

$y = r\sin\varphi\sin\theta$ $\qquad 0 \le \varphi \le \frac{\pi}{2}$ $\qquad r^2 = r\cos\varphi$

$z = r\cos\theta$ $\qquad 0 \le r \le ?$ $\qquad r = \cos\varphi$

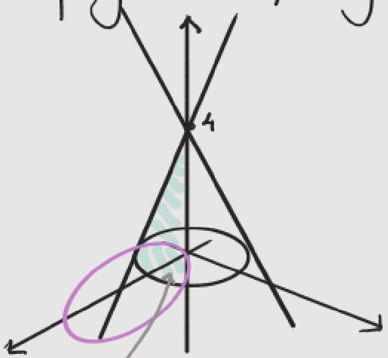
$J = r^2\sin\varphi$ $\qquad \rightsquigarrow\ 0 \le r \le \cos\varphi$



$$I = \iiint_T \sqrt{x^2+y^2+z^2}\,dxdydz = \int_{-\pi}^{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\cos\varphi} r\cdot r^2\sin\varphi\, dr\,d\varphi\,d\theta = \dots = \frac{\pi}{10}$$

□

(15.) Израчунати запремину тела $T = \{(x,y,z) \in \mathbb{R}^3 \mid (x-4)^2+y^2 \le 16,\ x^2+y^2 \le (z-4)^2,\ 0 \le z \le 4\}$



$z-4 = \sqrt{x^2+y^2}$ – горњи део конуса

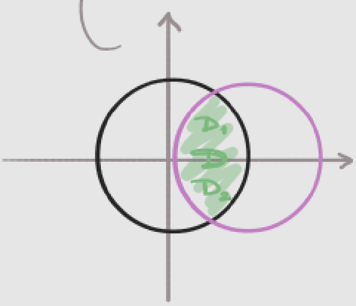
$z-4 = -\sqrt{x^2+y^2}$ – доњи део конуса

$(x-4)^2+y^2 = 16$ – цилиндар

$x^2+y^2 = (z-4)^2$ – конус, транслиран по z-оси

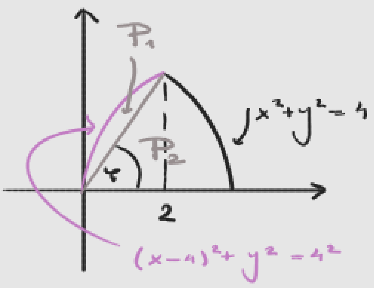
$z = 0 \rightsquigarrow x^2+y^2 = 4^2 \rightsquigarrow$ ово је пресек са Oxy равни

$$\rightsquigarrow \iiint_T dxdydz = \iint_D \int_0^{4-\sqrt{x^2+y^2}} dz\,dx\,dy = \iint_D \underbrace{\left(4-\sqrt{x^2+y^2}\right)}_{f(x,y)} dx\,dy = ⊛$$



Приметимо: $f(x,y) = f(x,-y)$

$\rightsquigarrow$ за тачку из D_1 и њој симетричну из D_2 вредност фје је иста $\rightsquigarrow ⊛ = 2\cdot\iint_{D_1}\left(4-\sqrt{x^2+y^2}\right) = \dots$

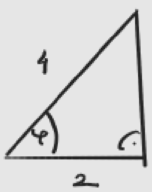


Уведимо поларне координате:

$x = r\cos\theta$

$y = r\sin\theta$, $J = r$

$$\iint_{D_1} = \iint_{P_1} + \iint_{P_2}$$



$\rightsquigarrow \cos\varphi = \frac{2}{4} = \frac{1}{2}$

$\Downarrow$

$\varphi = \frac{\pi}{3}$

Границе на P_1:

$\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$

$r^2 - 8r\cos\theta = 0$

$r = 8\cos\theta$

$\rightsquigarrow 0 \le r \le 8\cos\theta$

Границе на P_2:

$0 \le \theta \le \frac{\pi}{3}$

$0 \le r \le 4$

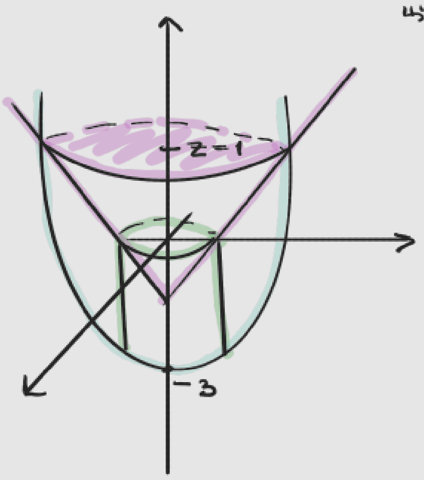
□

16. Наћи $V(T)$, T ограничено површима: $S_1: z = x^2+y^2-3$, $-3 \le z \le -2$, $S_2: x^2+y^2=1$, $-2 \le z \le 0$, $S_3: (z+1)^2 = x^2+y^2$, $0 \le z \le 1$, $S_4: z=1$

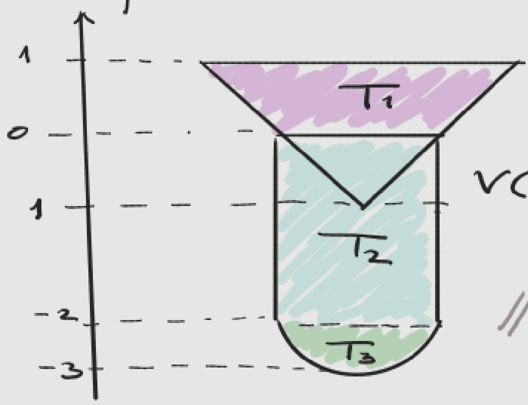
S_1 – параболоид (теме у $(0,0,-3)$)

S_2 – цилиндар

S_3 – конус (теме у $(0,0,-1)$)



Конус и параболоид се секу по кругу $x^2+y^2=4$ у равни $z=1$



$$V(T) = V(T_1)+V(T_2)+V(T_3)$$

// $V(\text{ваљка}) = B \cdot H$

$V(\text{купе}) = \frac{1}{3} B \cdot H$ //

$$V(T_1) = V(\text{веће купе}) - V(\text{мање купе})$$
$$= \frac{1}{3}\cdot\left(2^2\pi\cdot 2 - 1^2\cdot\pi\cdot 1\right) = \frac{7\pi}{3}$$

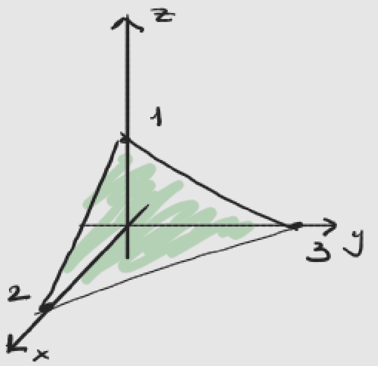
$$V(T_2) = 1\cdot\pi\cdot 2 = 2\pi$$

$$V(T_3) = \iint_D \Big(\int_{x^2+y^2-3}^{-2} dz\Big)\,dxdy = \iint_D (1-x^2-y^2)\,dxdy \overset{\text{поларне}}{=} \int_{-\pi}^{\pi}\int_0^1 (1-r^2)\cdot r\,dr\,d\theta = \dots = \frac{\pi}{2}$$

D – круг полупр. 1

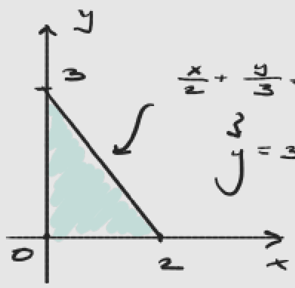
□

17. Израчунати $\iiint_T xyz\left(1-\frac{x}{2}-\frac{y}{3}-z\right)^n dxdydz$, $T = \left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; x,y,z \ge 0,\ \frac{x}{2}+\frac{y}{3}+z \le 1\right\}$



$\frac{x}{2}+\frac{y}{3}+z = 1$ – једначина равни

D:



$\frac{x}{2}+\frac{y}{3} = 1$

$y = 3\left(1-\frac{x}{2}\right)$

$$\iiint_T xyz\left(1-\frac{x}{2}-\frac{y}{3}-z\right)^n = \iint_D \Big(\int_0^{1-\frac{x}{2}-\frac{y}{3}} xyz\left(1-\frac{x}{2}-\frac{y}{3}-z\right)^n dz\Big)\,dxdy$$

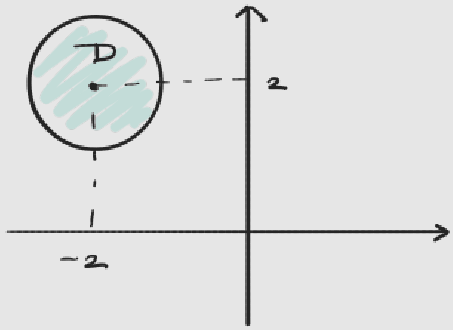
$$= \left\{\begin{aligned}&\text{смена: } t = 1-\frac{x}{2}-\frac{y}{3}-z\\ &dt = -dz\end{aligned}\right\} = \iint_D \Big(\int_0^{1-\frac{x}{2}-\frac{y}{3}} xy\left(1-\frac{x}{2}-\frac{y}{3}-t\right)\cdot t^n\,dt\Big)\,dxdy$$

$$= \iint_D xy \left(\int_0^{1-\frac{x}{2}-\frac{y}{3}} \left(1-\frac{x}{2}-\frac{y}{3}\right)t^n - t^{n+1}\, dt \right) dx\,dy = \iint_D xy \cdot \left(\frac{1}{n+1} - \frac{1}{n+2}\right)\left(1-\frac{x}{2}-\frac{y}{3}\right)^{n+2} dx\,dy$$

$$= \int_0^2 \left(\int_0^{3(1-\frac{x}{2})} xy \left(1-\frac{x}{2}-\frac{y}{3}\right)^{n+2} dx\,dy = \left\{ \begin{array}{c} t = 1-\frac{x}{2}-\frac{y}{3} \\ \vdots \end{array} \right\} = \dots = \frac{36}{(n+1)(n+2)\cdots(n+6)}\right.$$

$\square$

18. $V(T) = ?$, T огр. површима $z = x\cdot y$, $z = 0$, $x^2+y^2 = 4y - 4x - 7$



$x^2+y^2 = 4y - 4x - 7$

$x^2 + 4x + 4 - 4 + y^2 - 4y + 4 - 4 = -7$

$(x+2)^2 + (y-2)^2 = 1 \rightsquigarrow$ цилиндар

занимају нас само тачке из цилиндра, а ту је

$x\cdot y < 0 \rightsquigarrow z$ иде од $x\cdot y$ до 0!

$$\iiint_T dx\,dy\,dz = \iint_D \left(\int_{xy}^{0} dz\,dx\,dy \right) = \iint_D -xy\, dx\,dy = -\int_{-\pi}^{\pi} \int_0^1 (r\cos\theta - 2)(r\sin\theta + 2)\, r\, dr\, d\theta$$

$$= \dots = 4\pi$$

Овде је тешко наћи границе за стандардне поларне, па радимо померене.

$x = -2 + r\cos\theta$, $0 \le r \le 1$

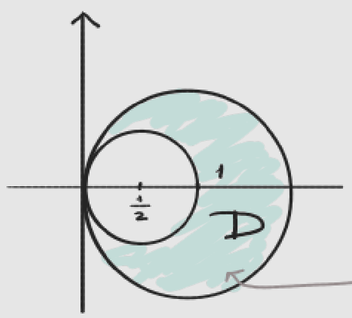
$y = 2 + r\sin\theta$, $-\pi \le \theta \le \pi$

$J = r$

$\square$

19. $V(T) = ?$, T огр. површима $x^2+y^2 = x$, $x^2+y^2 = 2x$, $z = x^2+y^2$, $z = 0$

$(x-\frac{1}{2})^2 + y^2 = (\frac{1}{2})^2$ $\qquad (x-1)^2 + y^2 = 1$ $\qquad$ параболоид



$$V(T) = \iiint_T dx\,dy\,dz = \iint_D \left(\int_0^{x^2+y^2} dz \right) dx\,dy = \iint_D (x^2+y^2)\, dx\,dy = I$$

ПОЛАРНЕ КООРДИНАТЕ НА D:

$x = r\cos\theta$, $J = r$

$y = r\sin\theta$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\underset{?}{\sqcup} \le r \le \underset{?}{\sqcup}$

(1): $r^2 = r\cos\theta \rightsquigarrow r = \cos\theta$ (за мањи круг)

(2) $r^2 = 2r\cos\theta \rightsquigarrow r = 2\cos\theta$ (за већи круг)

$\rightsquigarrow \cos\theta \le r \le 2\cos\theta$!

$$\rightsquigarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\cos\theta}^{2\cos\theta} r^2 \cdot r \, dr\, d\theta = \frac{15}{2} \int_0^{\frac{\pi}{2}} \cos^4\theta \, d\theta = \frac{15}{2} \int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2} \right)^2 d\theta = \dots = \frac{45}{32}\pi$$

◻

20. Наћи запремину тела T ограниченог површима $S_1: x^2+y^2-z^2=1$,

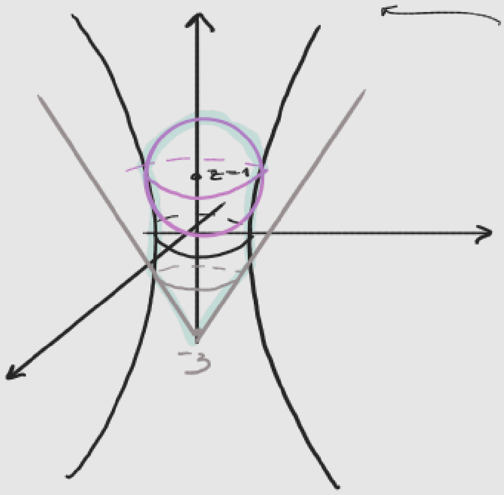
$S_2: \underbrace{x^2+y^2+z^2-2z=1}$, $z \geq 1$, $S_3: z = \sqrt{2(x^2+y^2)} - 3$

једнограни хиперболоид

$(x^2+y^2-z^2=-1$ – двограни$)$

$x^2+y^2+(z-1)^2=2$

горњи део конуса



пресек конуса и хиперболоида:

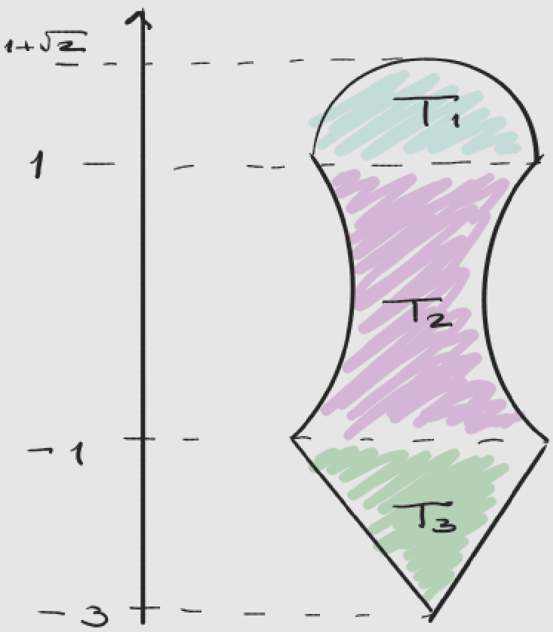
$$x^2+y^2 = z^2+1$$

$$z = \sqrt{2(x^2+y^2)} - 3$$

$$z+3 = \sqrt{2z^2+2} \quad /^2$$

$$z^2+6z+9 = 2z^2+2 \rightsquigarrow z^2-6z-7=0$$

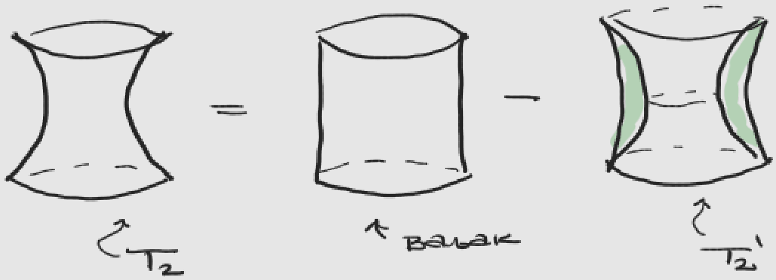
$$\rightsquigarrow z=-1 \lor z=7$$



половина запремине лопте

$$V(T_1) = \frac{1}{2} \cdot \frac{4}{3} r^3 \pi = \frac{4\sqrt{2}}{3}\pi$$

$$V(T_3) = \frac{1}{3} \cdot h \cdot r^2 \pi = \frac{4}{3}\pi$$



$$V(T_2') = \iint_D \left(\int_{-\sqrt{x^2+y^2-1}}^{\sqrt{x^2+y^2-1}} dz \right) dx\,dy = 2\iint_D \sqrt{x^2+y^2-1}\, dx\,dy = \dots$$

◻