* Ланчасто правило

Т: $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $g: \mathbb{R}^m \longrightarrow \mathbb{R}^k$ $\Longrightarrow$ $g\circ f: \mathbb{R}^n \longrightarrow \mathbb{R}^k$

Нека је $a \in \mathbb{R}^n$ т.д. f диф. у a и g диф. у $f(a)$. Тада је и $g\circ f$ диф. у a и важи:

$$d(g\circ f)(a) = \underbrace{\underbrace{d(g(f(a)))}_{k\times m} \cdot \underbrace{df(a)}_{m\times n}}_{k\times n \text{ матрица!}}$$

*** Специјално:** $k=1$ $\quad f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $g: \mathbb{R}^m \longrightarrow \mathbb{R}$ $\Longrightarrow$ $g\circ f: \mathbb{R}^n \longrightarrow \mathbb{R}$

Како рачунамо $\frac{\partial (g\circ f)}{\partial x_i}(a)$?

$$d(g\circ f)(a) = \left[\frac{\partial (g\circ f)}{\partial x_1}(a) \;\; \dots \;\; \frac{\partial (g\circ f)}{\partial x_n}(a)\right]$$

Са друге стране, по претходној Т. рачунамо:

$$d(g\circ f)(a) = dg(f(a))\cdot df(a) = \left[\frac{\partial g}{\partial y_1}(f(a)) \;\dots\; \frac{\partial g}{\partial y_m}(f(a))\right]_{1\times m} \cdot \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(a) & \dots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \dots & \frac{\partial f_m}{\partial x_n}(a) \end{bmatrix}_{m\times n}$$

$$= \left[\sum_{l=1}^{m} \frac{\partial g}{\partial y_l}(f(a))\cdot \frac{\partial f_l}{\partial x_1}(a) \;\;\dots\;\; \sum_{l=1}^{m} \frac{\partial g}{\partial y_l}(f(a))\cdot \frac{\partial f_l}{\partial x_n}(a)\right]_{1\times n}$$

Дакле: $$\boxed{\frac{\partial (g\circ f)}{\partial x_i}(a) = \sum_{l=1}^{m} \frac{\partial g}{\partial y_l}(f(a))\cdot \frac{\partial f_l}{\partial x_i}(a)}$$

* $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}$ + φ диференцијабилна

Дефинишемо тотални диференцијал функције φ на следећи начин:

$$d\varphi(a_1,\dots,a_n) := \frac{\partial \varphi}{\partial x_1}(a_1,\dots,a_n)\,dx_1 + \dots + \frac{\partial \varphi}{\partial x_n}(a_1,\dots,a_n)\,dx_n$$

(1.) Нека је $f: \mathbb{R} \longrightarrow \mathbb{R}$ и $u: \mathbb{R}^2 \longrightarrow \mathbb{R}$ дата са $u(x,y) = f(\sqrt{x^2+y^2})$. Наћи парцијалне изводе функције u у тачкама $(x,y) \neq (0,0)$.

$$\begin{tikzcd} (x,y) \arrow[r, mapsto, "g"] \arrow[rr, bend left, "u=f\circ g"] & \sqrt{x^2+y^2} \arrow[r, mapsto, "f"] & f(x^2+y^2) \end{tikzcd}$$

$\mathbb{R}^2 \qquad \mathbb{R} \qquad f: \mathbb{R} \to \mathbb{R}! \qquad \mathbb{R}$

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial (f\circ g)}{\partial x}(x,y) = f'(g(x,y))\cdot \frac{\partial g}{\partial x}(x,y) = f'(\sqrt{x^2+y^2})\cdot \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial u}{\partial y}(x,y) = \frac{\partial (f\circ g)}{\partial y}(x,y) = f'(g(x,y))\cdot \frac{\partial g}{\partial y}(x,y) = f'(\sqrt{x^2+y^2})\cdot \frac{y}{\sqrt{x^2+y^2}}$$

$\square$

(2.) $f:\mathbb{R}^3 \longrightarrow \mathbb{R}$ диференцијабилна функција и $\Psi:\mathbb{R}^2 \longrightarrow \mathbb{R}$ дата са $\Psi(x,y) = f(x^2+y^2,\ 2xy,\ x^2-y^2)$. Наћи тотални диференцијал функције Ψ.

$$\begin{tikzcd}
(x,y) \arrow[r, mapsto, "g"] \arrow[rr, bend left, "\Psi"] & (\underset{u(x,y)}{\underset{\shortparallel}{x^2+y^2}},\ \underset{v(x,y)}{\underset{\shortparallel}{2xy}},\ \underset{w(x,y)}{\underset{\shortparallel}{x^2-y^2}}) \arrow[r, mapsto, "f"] & f(u(x,y), v(x,y), w(x,y))
\end{tikzcd}$$

$g=(u,v,w)$ $\qquad\qquad$ $\Psi = f\circ g$

$d\Psi = \Psi'_x\, dx + \Psi'_y\, dy$ – по дефиницији

$$\Psi'_x(x,y) = \frac{\partial \Psi}{\partial x}(x,y) = f'_u \cdot u'_x + f'_v \cdot v'_x + f'_w \cdot w'_x$$
$$= f'_u \cdot 2x + f'_v \cdot 2y + f'_w \cdot 2x$$

$$\Psi'_y(x,y) = \frac{\partial \Psi}{\partial y}(x,y) = f'_u \cdot u'_y + f'_v \cdot v'_y + f'_w \cdot w'_y$$
$$= f'_u \cdot 2y + f'_v \cdot 2x + f'_w \cdot (-2y)$$

$\rightsquigarrow d\Psi = \Psi'_x\, dx + \Psi'_y\, dy$ $\square$

* пример: $f(t_1,t_2) = t_1 \operatorname{arctg}(e^{t_2}\cdot t_1) + 7$ $\qquad$ $h = f(u,v)$

$u(x,y,z) = x^2y$, $v(x,y,z) = z$

$$h(x,y,z) = f(u(x,y,z), v(x,y,z)) = f(x^2y, z) = x^2y \operatorname{arctg}(e^z\cdot x^2y) + 7$$

(1) рачунамо h'_x директно:

$$h'_x = 2xy\cdot\operatorname{arctg}(e^z\cdot x^2y) + x^2y\cdot\frac{1}{1+e^{2z}x^4y^2}\cdot 2xye^z = (*)$$

(2) рачунамо h'_x преко ланчастог правила:

$$h'_x = f'_{t_1}(u(x,y,z), v(x,y,z))\, u'_x(x,y,z) + f'_{t_2}(u(x,y,z), v(x,y,z))\cdot v'_x(x,y,z)$$

$$f'_{t_1}(t_1,t_2) = \operatorname{arctg}(e^{t_2}\cdot t_1) + t_1 e^{t_2}\frac{1}{1+e^{2t_2}t_1^2}$$

$$f'_{t_2}(t_1,t_2) = e^{t_2} t_1^2 \frac{1}{1+e^{2t_2}t_1^2}$$

$$\rightsquigarrow h'_x = \left(\operatorname{arctg}(e^z\cdot x^2y) + x^2ye^z\frac{1}{1+e^{2z}x^4y^2}\right)\cdot 2xy + e^z x^4y^2\frac{1}{1+e^{2z}x^4y^2}\cdot 0$$
$$= 2xy \operatorname{arctg}(e^z\cdot x^2y) + x^2ye^z\cdot\frac{1}{1+e^{2z}x^4y^2}\cdot 2xy = (**)$$

Видимо да добијамо исто, тј. $(*) = (**)$!

* ТРАНСФОРМАЦИЈЕ КООРДИНАТА У ДИФ. ЈЕДНАЧИНАМА

(3.) Трансформисати једначину $x^2 f''_{xx} - y^2 f''_{yy} - 2y f'_y = 0$ узимајући $u = xy$, $v = \frac{y}{x}$ за нове независне променљиве

За почетак, желимо да изразимо f'_x преко f'_u и f'_v

нпр: $f(x,y) = xy + \left(\frac{y}{x}\right)^2$

$f'_x = y + 2\frac{y}{x}\cdot\left(-\frac{y}{x^2}\right) \xrightarrow{\text{преко } u,v} f(x,y) = \tilde{f}(u,v) = u + v^2$, $\quad f'_u = 1$

$$f'_x = f'_u\cdot u'_x + f'_v\cdot v'_x = f'_u\cdot y + f'_v\cdot\left(-\frac{y}{x^2}\right)$$

f'_u је такође ф-ја по u,v!

I начин: $f''_{xx} = (f'_x)_x = (f'_u\cdot u'_x + f'_v\cdot v'_x)'_x = (f'_u)_x\cdot u'_x + f'_u\cdot u''_{xx} + (f'_v)_x\cdot v'_x + f'_v\cdot v''_{xx}$

$(f'_u)_x = f''_{uu}\cdot u'_x + f''_{uv}\cdot v'_x \qquad (f'_v)_x = f''_{vu}\cdot u'_x + f''_{vv}\cdot v'_x$

$$= f''_{uu}\cdot(u'_x)^2 + f''_{uv}\cdot u'_x\cdot v'_x + f'_u\cdot u''_{xx} + f''_{vv}(v'_x)^2 + f''_{vu}\cdot v'_x\cdot u'_x + f'_v\cdot v''_{xx}$$

$$= f''_{uu}\cdot y^2 + f''_{uv}\left(-\frac{y^2}{x^2}\right) + f'_u\cdot 0 + f''_{vv}\frac{y^2}{x^4} + f''_{vu}\left(-\frac{y^2}{x^2}\right) + f'_v\frac{2y}{x^3}$$

II начин: $f''_{xx} = \left(f'_u\cdot y + f'_v\left(-\frac{y}{x^2}\right)\right)'_x = y\cdot(f'_u)'_x + \frac{2y}{x^3}f'_v - \frac{y}{x^2}(f'_v)'_x$

$$= y\cdot(f''_{uu}\cdot u'_x + f''_{uv}\cdot v'_x) - \frac{y}{x^2}(f''_{vu}\cdot u'_x + f''_{vv}\cdot v'_x) + \frac{2y}{x^3}f'_v$$

$$= y^2 f''_{uu} - \frac{y^2}{x^2}f''_{uv} - \frac{y^2}{x^2}f''_{vu} + \frac{y^2}{x^4}f''_{vv} + \frac{2y}{x^3}f'_v$$

НАРАВНО, ДОБИЈАМО ИСТО ШТО И У I

На сличан начин се добија $f'_y = f'_u\cdot u'_y + f'_v\cdot v'_y = f'_u\cdot x + \frac{1}{x}\cdot f'_v$

и $f''_{yy} = f''_{uu}\cdot x^2 + f''_{uv}\cdot 1 + f''_{vu} + f''_{vv}\cdot\frac{1}{x^2}$

Вратимо се у почетну једначину:

$$x^2 f''_{xx} - y^2 f''_{yy} - 2y f'_y = 0$$

$$\cancel{x^2y^2 f''_{uu}} - y^2 f''_{uv} - y^2 f''_{vu} + \cancel{\frac{y^2}{x^2}} f''_{vv} + \cancel{\frac{2y}{x} f'_v} - \cancel{x^2y^2 f''_{uu}} - y^2 f''_{uv} - y^2 f''_{vu} - \cancel{\frac{y^2}{x^2}} f''_{vv}$$

$$-2yx f'_u - \cancel{2\frac{y}{x} f'_v} = 0$$

$$-2y^2(f''_{vu} + f''_{uv}) - 2xy f'_u = 0 \;/ : (-2)$$

$$y^2(f''_{vu} + f''_{uv}) + xy f'_u = 0$$

$$u = xy, \quad v = \frac{y}{x} \rightsquigarrow y^2 = uv$$

$$\Rightarrow -uv(f''_{vu} + f''_{uv}) - u f'_u = 0$$

■

4. Увести смену $x = \operatorname{tg} t$ ($t = \operatorname{arctg} x$) у једначину $(1+x^2)^2 y'' + 2x(1+x^2)y' + y = 0$, ако је $y = y(x)$.

У ће постати функција по t! $(\tilde{y}(t) = y(x(t)))$

$$y'_t = \frac{\partial y}{\partial t} = \frac{\partial y}{\partial x}\cdot\frac{\partial x}{\partial t} = y'\cdot\frac{1}{\cos^2 t} \Rightarrow y' = \cos^2 t\cdot y'_t$$

$$y'' = (y')'_x = (\cos^2 t \cdot y'_t)'_x = (\cos^2 t \cdot y'_t)'_t \cdot \frac{\partial t}{\partial x} = \left(2\cos t\,(-\sin t)\cdot y'_t + \cos^2 t\, y''_{tt}\right) \cdot \underbrace{\cos^2 t}_{\frac{\partial t}{\partial x}}$$

$$= -2\cos^3 t \sin t \cdot y'_t + \cos^4 t\, y''_{tt}$$

Вратимо у почетну једначину:

$$(1+tg^2 t)^2 \cdot \left(-2\cos^3 t \sin t \cdot y'_t + \cos^4 t\, y''_{tt}\right) + 2tg\,t\,\cancel{(1+tg^2 t)} \cdot \cancel{\cos^2 t} \cdot y'_t + y = 0$$

$$\frac{1}{\cos^4 t}\,\cancel{(-2\cos^3 t \sin t\, y'_t)} + y''_{tt} + \cancel{2tg\,t \cdot y'_t} + y = 0$$

$$\rightsquigarrow y''_{tt} + y = 0.$$

$\blacksquare$

⑤ Трансформисати израз $u''_{xx} + u''_{yy}$ помоћу поларних координата $x = r\cos\theta \ (1)$, $y = r\sin\theta \ (2)$, $r > 0$

Можемо на 2 начина:

$$u'_x = u'_r \cdot r'_x + u'_\theta \cdot \theta'_x \quad , \quad r = r(x,y), \ \theta = \theta(x,y)$$

I: $(1)'_x: \ 1 = r'_x \cos\theta + r(-\sin\theta)\cdot\theta'_x$

$(2)'_x: \ 0 = r'_x \sin\theta + r\cos\theta \cdot \theta'_x \quad / \cdot \left(-\frac{\cos\theta}{\sin\theta}\right)$ + Решимо систем

$$\rightsquigarrow 1 = -r\sin\theta\,\theta'_x - r\,\frac{\cos^2\theta}{\sin\theta}\,\theta'_x$$

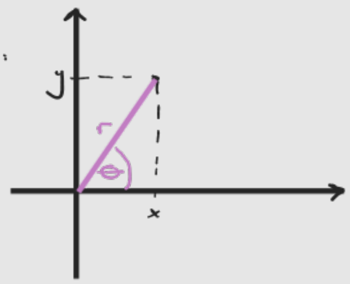
$$1 = -r\theta'_x\left(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta}\right) = -\frac{r}{\sin\theta}\cdot\theta'_x \implies \theta'_x = -\frac{\sin\theta}{r}$$

$$\rightsquigarrow 0 = r'_x \sin\theta + \cancel{r}\cos\theta \cdot \left(-\frac{\sin\theta}{\cancel{r}}\right)$$

$$0 = r'_x \sin\theta - \sin\theta\cos\theta$$

$$\implies r'_x = \cos\theta$$

II:



$$r^2 = x^2 + y^2 \quad / \frac{\partial}{\partial x}$$

$$2r\cdot r'_x = 2x \implies r'_x = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{y}{x} = tg\,\theta \implies \theta = arctg\,\frac{y}{x} \quad / \frac{\partial}{\partial x}$$

$$\theta'_x = \frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right)$$

$$= \frac{1}{x^2+y^2}\cdot(-y) = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

$$\rightsquigarrow u'_x = u'_r \cdot \cos\theta - u'_\theta \cdot \frac{\sin\theta}{r}$$

$$u''_{xx} = (u'_x)'_x = \left(u'_r \cdot \cos\theta - u'_\theta \cdot \frac{\sin\theta}{r}\right)'_x = \ldots$$