Puasonov proces

12. $X(t)$ - homogen Puasonov proces

λ - intenzitet procesa

$0 \leq s < t$

a) $p\{X(t)=n+k \mid X(s)=n\}$

(nezavisnost prirastaja)

$$= p\{X(t)-X(s)=k\} = \frac{e^{-\lambda(t-s)} \cdot (\lambda(t-s))^k}{k!}$$

b) $$p\{X(s)=n \mid X(t)=n+k\} = \frac{1}{\frac{e^{-\lambda t} \cdot (\lambda t)^{n+k}}{(n+k)!}} \cdot p\{X(s)=n,\ X(t)=n+k\}$$

$$= \frac{(n+k)!}{e^{-\lambda t} \cdot (\lambda t)^{n+k}} \cdot p\{X(s)=n\} \cdot p\{X(t)=n+k \mid X(s)=n\}$$

$$= \frac{(n+k)!}{e^{-\lambda t} \cdot (\lambda t)^{n+k}} \cdot \frac{e^{-\lambda s} \cdot (\lambda s)^n}{n!} \cdot \frac{e^{-\lambda(t-s)} \cdot (\lambda(t-s))^k}{k!}$$

$$= \frac{(n+k)!}{n! \cdot k!} \cdot \frac{s^n \cdot (t-s)^k}{t^{n+k}} = \binom{n+k}{n} \cdot \left(\frac{s}{t}\right)^n \cdot \left(1-\frac{s}{t}\right)^k \sim \boxed{\mathcal{B}\left(n+k, \frac{s}{t}\right)}$$

v) $E(X(u)-X(s) \mid X(t)=n)$, $0 \leq s < t < u$

$$= E(\underbrace{X(u)-X(t)}_{\mathcal{P}(\lambda(u-t))} \mid X(t)=n) + E(X(t)-\underbrace{X(s)}_{\mathcal{B}(n, \frac{s}{t})} \mid X(t)=n)$$

$$= \lambda(u-t) + n - n \cdot \frac{s}{t}$$

13. $N(t)$ - broj klijenata

$\lambda = 4$ (merimo vreme po satu)

a) $N(1) = 2$

$$p\{\underbrace{N(\tfrac{1}{3})}_{\mathcal{B}(2,\frac{1}{3})} = 2 \mid N(1) = 2\} = (\tfrac{1}{3})^2 = \tfrac{1}{9}$$

b) 15% klijenata pravna lica

$\Rightarrow N_1(t)$ Puasonov proces sa int. $\lambda_1 = \lambda \cdot 0.15 = 0.6$

$$N(t) = \underbrace{N_1(t)}_{\text{pravna lica}} + \underbrace{N_2(t)}_{\text{Fizička lica}}$$

$$p\{N_1(8) = 0\} = e^{-8\cdot\lambda_1} = e^{-4.8}$$

Alternativno,

$$p\{N_1(8) = 0\} = p\{T_1 > 8\} = e^{-8\cdot\lambda_1} = e^{-4.8}$$

T_1 - vreme dolaska prvog klijenta (pravnog lica) $\sim \mathcal{E}(\lambda_1)$

Zbog odsustva memorije, vreme čekanja na prvog klijenta, tj. pravno lice, počevši od 12h, je ekvivalentno čekanju prvog klijenta pri otvaranju banke.

$$ET_1 = \frac{1}{\lambda_1} = \frac{1}{0.6} = 100 \text{ min}$$

Očekivano vreme dolaska je, dakle, 13:40

14. $N_J(t)$ - broj automobila sa juga, intenzitet a

$N_S(t)$ - broj automobila sa severa, intenzitet b

Pri dolasku taksija sa severa se vreme resetuje i možemo da smatramo da kreće od nule. Zbog toga se problem svodi na izvođenje raspodele $N_J(T_{S1})$; T_{S1} - dolazak prvog taksija sa severa

$T_{S1} \sim \mathcal{E}(b)$

$$p\{N_J(T_{S1}) = k\} = \int_0^{+\infty} b e^{-bt} \cdot p\{N_J(T_{S1}) = k \mid T_{S1} = t\}\, dt$$

$$= \int_0^{+\infty} b e^{-bt} \cdot \frac{e^{-at} \cdot (at)^k}{k!}\, dt$$

$$= \frac{a^k \cdot b}{(a+b)^{k+1}} \cdot \int_0^{+\infty} \underbrace{\frac{e^{-(a+b)t} \cdot (a+b)^{k+1} \cdot t^k}{k!}}_{f_{\Gamma}(k,\,a+b)}\, dt$$

$$= \left(\frac{a}{a+b}\right)^k \cdot \frac{b}{a+b}$$

$$\Rightarrow N_J(T_{S1}) \sim \mathcal{G}\left(\frac{b}{a+b}\right)$$

II način:

Drugi pristup podrazumeva da resetujemo vreme svaki put kada dođe auto sa juga, i intuitivnije se dolazi do geometrijske raspodele.

pr. $p\{N_J(T_{S1}) = 0\} = p\{T_{S1} < T_{J1}\} = \dfrac{b}{a+b}$

$$p\{N_J(T_{S1}) = 1\} = p\{T_{S1} > T_{J1},\ T_{S1} < T_{J2}\}$$

$$= p\{T_{S1} > T_{J1}\} \cdot p\{T_{S1} < T_{J2} \mid T_{S1} > T_{J1}\}$$

$$= \frac{a}{a+b} \cdot p\{\underbrace{T_{S1} - T_{J1}}_{\mathcal{E}(b)} < \underbrace{T_{J2} - T_{J1}}_{\mathcal{E}(a)} \mid T_{S1} > T_{J1}\}$$

$$= \frac{a}{a+b} \cdot \frac{b}{a+b}$$

$\vdots$

$$p\{N_J(T_{S1}) = k\} = p\{T_{S1} > T_{J1}\}\, p\{T_{S1} > T_{J2} \mid T_{S1} > T_{J1}\} \cdots$$
$$\cdot\, p\{T_{S1} < T_{Jk+1} \mid T_{S1} > T_{Jk}\}$$
$$= \left(\frac{a}{a+b}\right)^k \cdot \frac{b}{a+b}$$

15. W - vreme čekanja

T_i - trenuci dolaska automobila, mereni od trenutka dolaska deteta

Ako $T_1 \geq t$, $W = 0$,

inače ako $T_2 - T_1 \geq t$, $W = T_2 - t$,

inače ako $T_3 - T_2 \geq t$, $W = T_3 - t$,

...

Izračunaćemo EW rekurzivno, resetujući vreme pri svakom dolasku.

$$EW = p\{T_1 \geq t\} \cdot 0 + p\{T_1 < t\} \cdot E(W \mid T_1 < t)$$

$$= (1 - e^{-\lambda t}) \cdot E(W - T_1 + T_1 \mid T_1 < t)$$

$$= (1 - e^{-\lambda t}) \cdot (E(T_1 \mid T_1 < t) + EW)$$ ← Kada dođe prvi automobil, devojčica kao da čeka ispočetka

$$EW \cdot e^{-\lambda t} = (1 - e^{-\lambda t}) \cdot \int_0^t \lambda e^{-\lambda x} \cdot x \, dx$$

$$EW = (e^{\lambda t} - 1) \cdot \frac{1}{\lambda} \int_0^{\lambda t} e^{-y} y \, dy$$

$$= \frac{e^{\lambda t} - 1}{\lambda} \left(-\int_0^{\lambda t} y \cdot d e^{-y}\right) = \frac{e^{\lambda t} - 1}{\lambda} \cdot \left(-y e^{-y}\Big|_0^{\lambda t} + \int_0^{\lambda t} e^{-y} dy\right)$$

$$= \frac{e^{\lambda t} - 1}{\lambda} \cdot (-\lambda t e^{-\lambda t} + 1 - e^{-\lambda t})$$

16. Računamo vreme od dolaska nultog klijenta, tj od 7:30, u minutima.

Informacije koje imamo:

$T_1 \leq 4$, $T_2 \leq 10$, $T_3 = 15$

$$f_{T_1 \mid T_1 \leq 4,\, T_2 \leq 10,\, T_3 = 15}(x) = \frac{\overset{\text{dolazak prvog klijenta}}{\lambda e^{-\lambda x}} \cdot \int_x^{10} \overset{\text{dolazak drugog klijenta}}{\lambda e^{-\lambda(t-x)}} \cdot \overset{\text{dolazak trećeg klijenta}}{\lambda e^{-\lambda(15-t)}}\, dt}{\underbrace{\int_0^4 \lambda e^{-\lambda x} \cdot \int_x^{10} \lambda e^{-\lambda(t-x)} \cdot \lambda e^{-\lambda(15-t)}\, dt\, dx}_{= C \text{ (ne zavisi od } x)}}$$

$$= \frac{1}{C} \cdot \lambda e^{-\lambda x} \cdot \int_x^{10} \lambda^2 e^{-\lambda(15-x)}\, dt$$

$$= \frac{1}{C} \cdot \lambda^3 \cdot e^{-\lambda x} \cdot (10-x)\, e^{-\lambda(15-x)}$$

$$= \underbrace{\frac{1}{C} \cdot \lambda^3 e^{-15\lambda}}_{= C_1} \cdot (10-x)$$

$$f_{T_1 \mid T_1 \leq 4,\, T_2 \leq 10,\, T_3 = 15}(x) = C_1 \cdot (10-x), \quad x \in [0, 4]$$

$$C_1 \cdot \int_0^4 (10-x)\, dx = 1$$

$$C_1 \cdot \left(40 - \frac{4^2}{2}\right) = 1 \qquad C_1 = \frac{1}{32}$$

$$E(T_1 \mid T_1 \leq 4,\, T_2 \leq 10,\, T_3 = 15) = \int_0^4 \frac{10-x}{32} \cdot x\, dx$$

$$= \frac{10}{32} \cdot \frac{4^2}{2} - \frac{1}{32} \cdot \frac{4^3}{3} = \frac{5}{2} - \frac{2}{3} = \frac{11}{6}$$

Očekivano vreme: 7:31:50