Osobine eksponencijalne raspodele

1° Odsustvo memorije

$X \sim \mathcal{E}(\lambda)$, $t > 0$

$$\boxed{X - t \mid X > t \sim X}$$

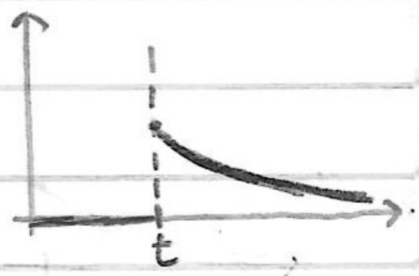
Dokaz: $u > 0$ $F_{X-t\mid X>t}(u) = P\{X - t \le u \mid X > t\}$

$$= 1 - P\{X > u + t \mid X > t\}$$

$$= 1 - \frac{P\{X > u+t\}}{P\{X > t\}}$$

$$= 1 - \frac{e^{-\lambda(u+t)}}{e^{-\lambda t}}$$

$$= 1 - e^{-\lambda u} = F_X(u)$$



$\Rightarrow X - t \mid X > t \sim X$

Posledica:

$X \mid X > t \sim X + t \leftarrow$ pomereno eksp. $f_X(x) = \lambda e^{-\lambda(x-t)}$, $x \ge t$

2° Jako odsustvo memorije

$Y > 0$ slučajna veličina, nezavisna od X

$$\boxed{X - Y \mid X > Y \sim X}$$

Dokaz: $P\{X - Y \le t \mid X > Y\} = 1 - P\{X > Y + t \mid X > Y\}$

$$= 1 - \frac{P\{X > Y + t\}}{P\{X > Y\}} = 1 - \frac{\int_0^{+\infty} P\{X > y + t\}\, dF_Y(y)}{\int_0^{+\infty} P\{X > y\}\, dF_Y(y)}$$

$$= 1 - \frac{\int_0^{+\infty} e^{-\lambda(y+t)} dF_Y(y)}{\int_0^{+\infty} e^{-\lambda y} \cdot dF_Y(y)} = 1 - \frac{e^{-\lambda t} \cdot \cancel{\int_0^{+\infty} e^{-\lambda y} dF_Y(y)}}{\cancel{\int_0^{+\infty} e^{-\lambda y} dF_Y(y)}}$$

$$= 1 - e^{-\lambda t}$$

$$\Rightarrow X - Y \mid X > Y \sim \mathcal{E}(\lambda) \sim X.$$

Posledica:

$$X \mid X > Y \sim X + (Y \mid X > Y)$$

3° $X_1 \sim \mathcal{E}(\lambda_1)$, $X_2 \sim \mathcal{E}(\lambda_2)$, nezavisne

$$\boxed{\min\{X_1, X_2\} \sim \mathcal{E}(\lambda_1 + \lambda_2)}$$

$$p\{\min\{X_1, X_2\} \le t\} = 1 - p\{\min\{X_1, X_2\} > t\}$$

$$= 1 - p\{X_1 > t, X_2 > t\}$$

$$= 1 - e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} = 1 - e^{-(\lambda_1 + \lambda_2)t}$$

Opštije: $X_1, X_2, \dots, X_n$ nezavisne, $X_i \sim \mathcal{E}(\lambda_i)$

$$\min\{X_1, X_2, \dots, X_n\} \sim \mathcal{E}(\lambda_1 + \lambda_2 + \dots + \lambda_n)$$

4° $X_1 \sim \mathcal{E}(\lambda_1)$, $X_2 \sim \mathcal{E}(\lambda_2)$, nezavisne

$$\boxed{p\{X_1 < X_2\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}}$$

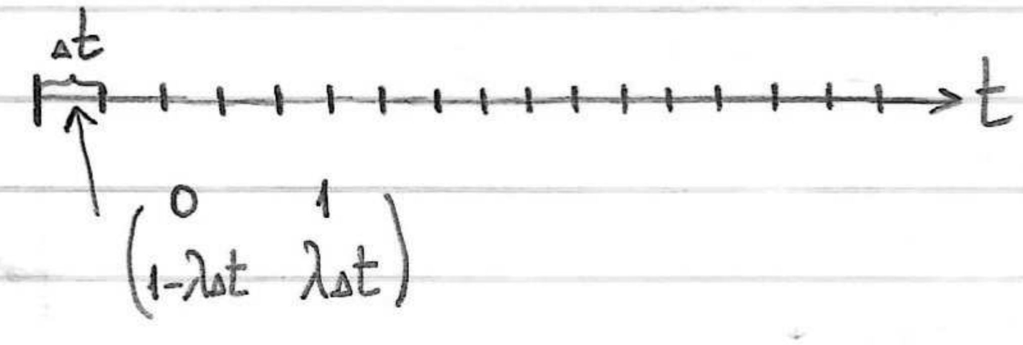
$$p\{X_1 < X_2\} = \int_0^{+\infty} \lambda_1 e^{-\lambda_1 x} p\{X_2 > x\} dx$$

$$= \int_0^{+\infty} \lambda_1 e^{-\lambda_1 x} \cdot e^{-\lambda_2 x} dx = \frac{\lambda_1}{\lambda_1 + \lambda_2} \overbrace{\int_0^{+\infty} (\lambda_1 + \lambda_2) e^{-(\lambda_1 + \lambda_2)x} dx}^{=1}$$

Opštije: $X_i \sim \mathcal{E}(\lambda_i)$ nezavisne

$$p\{X_k = \min\{X_1, \dots, X_n\}\} = \frac{\lambda_k}{\lambda_1 + \dots + \lambda_n}$$

Mnoge ove osobine se lakše vide, tj. postaju intuitivnije, kada se eksponencijalna raspodela posmatra u kontekstu pozadinskog homogenog Puasonovog procesa:



Ako vreme podelimo u beskonačno male periode, Puasonov proces se dobija pretpostavkom da u svakom periodu trajanja Δt, događaj može da se desi sa verovatnoćom $\lambda \cdot \Delta t$, nezavisno od prošlosti i budućnosti.
Odsustvo memorije tada sledi iz te nezavisnosti:
Činjenica da se nijedan događaj nije desio do trenutka t ne nosi nikakvu informaciju o budućnosti, pa je kao da smo resetovali vreme.
Raspodela minimuma dve nezavisne eksponencijalne sledi iz posmatranja dva nezavisna procesa - Verovatnoća da se u periodu Δt desi događaj u bar jednom od njih je $1-(1-\lambda_1\Delta t)\cdot(1-\lambda_2\Delta t) = (\lambda_1+\lambda_2)\Delta t + o(\Delta t)$
Slično, verovatnoća da je jedna eksp. manja od druge se dobija posmatrajući period Δt, i uslovnu verovatnoću da je događaj potekao iz prvog procesa, ako znamo da se desio događaj u jednom od dva procesa:

$$\frac{\lambda_1\Delta t}{\lambda_1\Delta t+\lambda_2\Delta t} = \frac{\lambda_1}{\lambda_1+\lambda_2}$$

2. T - trajanje sijalice , $T \sim \mathcal{E}(\lambda)$

$ET = 10h \Rightarrow \lambda = \frac{1}{10h}$

t - trenutak ulaska osobe X

Sijalica i dalje gori $\Rightarrow T > t$

$$p\{T > 5h + t \mid T > t\} \overset{OM}{=} p\{T > 5h\} = e^{-\frac{5}{10}} = e^{-\frac{1}{2}}$$

3. $T_A \sim \mathcal{E}(\lambda_A)$, $T_B \sim \mathcal{E}(\lambda_B)$

a) $E(\min\{T_A, T_B\}) = \frac{1}{\lambda_A + \lambda_B}$

b) $p\{T_A < T_B\} = \frac{\lambda_A}{\lambda_A + \lambda_B}$

4. $X_1, X_2 \sim \mathcal{E}(a)$

$Y_1, Y_2 \sim \mathcal{E}(4a)$

$X = \min\{X_1, X_2\} \sim \mathcal{E}(2a)$

$Y = Y_1 + Y_2 \sim \Gamma(2, 4a)$

$p\{X < Y\} = ?$

Nije teško pešački izračunati verovatnoću da je eksponencijalna manja od game, ali možemo i da koristimo jako odsustvo memorije.

$$p\{X < Y\} = 1 - p\{X > Y\} = 1 - p\{X > Y_1 + Y_2 \mid X > Y_1\} \cdot p\{X > Y_1\}$$

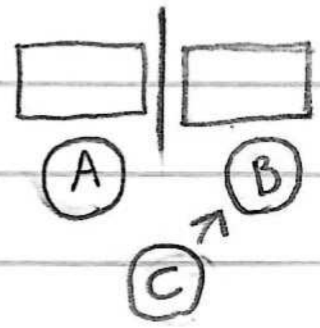
$$= 1 - p\{X - Y_1 > Y_2 \mid X > Y_1\} \cdot p\{X > Y_1\}$$

$$= 1 - p\{X > Y_2\} \cdot p\{X > Y_1\} = 1 - \frac{4a}{4a + 2a} \cdot \frac{4a}{4a + 2a} = \frac{5}{9}$$

5. $X \sim \mathcal{E}(\lambda)$

$X \mid X > 1 \sim X + 1$ OM

$\Rightarrow E(X^2 \mid X > 1) = E(X+1)^2$

6. 

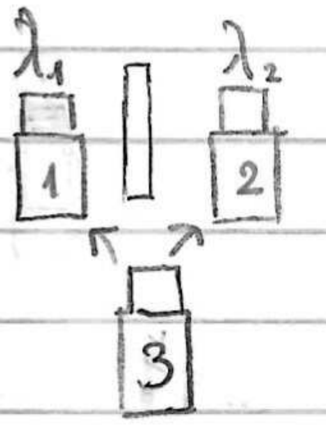
$T_X \sim \mathcal{E}(\lambda)$

T_X - vreme za šalterom osobe X

$$p\{T_A > T_B + T_C\} = p\{T_A > T_B\} \cdot p\{T_A > T_B + T_C \mid T_A > T_B\}$$

$$= \frac{\lambda}{\lambda + \lambda} \cdot p\{T_A - T_B > T_C \mid T_A > T_B\}$$

$$= \frac{1}{2} \cdot p\{T_A > T_C\} = \frac{1}{4}$$

7. 

T_i - trajanje carinjenja vozila i

$T_1 \sim \mathcal{E}(\lambda_1)$

$T_2 \sim \mathcal{E}(\lambda_2)$

T_u - ukupno čekanje šlepera, tj. vozila 3

$T_u = \min\{T_1, T_2\} + T_3$

$$ET_u = p\{T_1 < T_2\} \cdot E(T_u \mid T_1 < T_2) + p\{T_2 < T_1\} \cdot E(T_u \mid T_2 < T_1)$$

opslužuje ga terminal 1

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2}\left(E(T_3 \mid T_1 < T_2) + E(\min\{T_1, T_2\} \mid T_1 < T_2)\right)$$

opslužuje ga terminal 2

$$+ \frac{\lambda_2}{\lambda_1 + \lambda_2}\left(E(T_3 \mid T_2 < T_1) + E(\min\{T_1, T_2\} \mid T_2 < T_1)\right)$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_1 + \lambda_2}\right) + \frac{\lambda_2}{\lambda_1 + \lambda_2}\left(\frac{1}{\lambda_2} + \frac{1}{\lambda_1 + \lambda_2}\right) = \frac{3}{\lambda_1 + \lambda_2}$$

* Raspodela minimuma se ne menja sa znanjem odakle potiče

$$\boxed{\min\{T_1, T_2\} \mid T_1 < T_2 \sim \min\{T_1, T_2\}}$$

$$P\{\min(T_1,T_2) \le t \mid T_1 < T_2\} = P\{T_1 \le t \mid T_1 < T_2\} = \frac{P\{T_1 \le t,\, T_1 < T_2\}}{P\{T_1 < T_2\}}$$

$$= \frac{\lambda_1 + \lambda_2}{\cancel{\lambda_1}} \cdot \int_0^t \cancel{\lambda_1} e^{-\lambda_1 x} \cdot P\{T_2 > x\}\, dx$$

$$= (\lambda_1 + \lambda_2) \int_0^t e^{-(\lambda_1+\lambda_2)x}\, dx = 1 - e^{-(\lambda_1+\lambda_2)x}$$

8. $T_1 \sim \mathcal{E}(\lambda_1) \quad T_2 \sim \mathcal{E}(\lambda_2)$

$$P\{T_1 < T_2 + t\} = P\{T_1 < t\} + P\{T_1 - t < T_2 \mid T_1 > t\} \cdot P\{T_1 > t\}$$

$$= 1 - e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot e^{-\lambda_1 t}$$

9.

μ_1	μ_2
$\boxed{S_1}$	$\boxed{S_2}$
Z	X
	Y

T_X - trajanje opsluživanja X

T_Y - trajanje opsl. Y

T_{Z1}, T_{Z2} - trajanja opsluživanja Z na $S_{1,2}$

$T_X, T_Y, T_{Z2} \sim \mathcal{E}(\mu_2)$

$T_{Z1} \sim \mathcal{E}(\mu_1)$

a) $P_X = P\{T_{Z1} < T_X\} = \frac{\mu_1}{\mu_1 + \mu_2}$

b) $P_Y = P\{T_{Z1} < T_X + T_Y\} = P_X + P\{T_{Z1} - T_X < T_Y \mid T_{Z1} > T_X\} \cdot (1 - P_X)$

$$(\text{JOM}) \quad = \frac{\mu_1}{\mu_1+\mu_2} + P\{T_{Z1} < T_Y\} \cdot \frac{\mu_2}{\mu_1+\mu_2} = \frac{\mu_1}{\mu_1+\mu_2}\left(1 + \frac{\mu_2}{\mu_1+\mu_2}\right)$$

v) $T_u = T_o + T_č$

T_o - vreme opsluživanja Z , $ET_o = ET_{z_1} + ET_{z_2} = \frac{1}{\mu_1} + \frac{1}{\mu_2}$.

$T_č$ - vreme čekanja u redu

$$ET_č = P_x \cdot E(T_č \mid T_{z_1} < T_x) + (P_Y - P_x) E(T_č \mid T_x < T_{z_1} < T_x + T_Y)$$

$$+ (1 - P_Y) \cdot E(T_č \mid T_{z_1} > T_x + T_Y)$$

$$E(T_č \mid T_{z_1} < T_x) = E(T_x + T_Y) = \frac{2}{\mu_2}$$

$\uparrow$

Z zatiče X i Y u redu za S_2 ispred sebe, zbog JOM se vreme „resetuje", kao da se X opslužuje ispočetka.

$E(T_č \mid T_x < T_{z_1} < T_x + T_Y) = E(T_Y) = \frac{1}{\mu_2}$, Z čeka da Y završi

$E(T_č \mid T_{z_1} > T_x + T_Y) = 0$, Z ne čeka

$$ET_u = \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{\mu_1}{\mu_1 + \mu_2} \cdot \frac{2}{\mu_2} + \frac{\mu_1}{\mu_1 + \mu_2}\left(1 + \frac{\mu_2}{\mu_1 + \mu_2}\right) \cdot \frac{1}{\mu_2}$$

* $X_1 \sim \mathcal{E}(\lambda_1)$, $X_2 \sim \mathcal{E}(\lambda_2)$

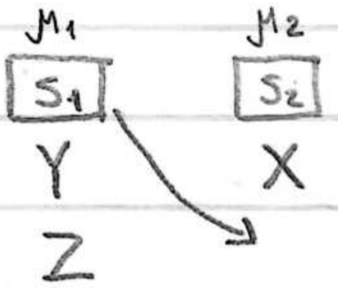
$$- E(\max\{X_1, X_2\}) = p\{X_1 < X_2\} \cdot E(X_2 \mid X_1 < X_2) + p\{X_1 > X_2\} \cdot E(X_1 \mid X_1 > X_2)$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot E(X_2 + \overbrace{(X_1 \mid X_1 < X_2)}^{\sim \min\{X_1, X_2\}}) + \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot E(X_1 + \overbrace{(X_2 \mid X_2 < X_1)}^{\sim \min\{X_1, X_2\}})$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot E(X_2) + \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot E(X_1) + E(\min\{X_1, X_2\})$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot \frac{1}{\lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot \frac{1}{\lambda_1} + \frac{1}{\lambda_1 + \lambda_2} = \frac{1}{\lambda_1 + \lambda_2}\left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}\right)$$

* $\lambda_1 = \lambda_2 = \lambda$ $\quad E(\max\{X_1, X_2\}) = \frac{3}{2} \cdot \frac{1}{\lambda} = \frac{3}{2} \cdot E(X_1)$

10. 

I način:

$T_u = T_o + T_č$ $\qquad ET_o = \frac{1}{\mu_1} + \frac{1}{\mu_2}$

$T_č = T_{č_1} + T_{č_2}$

$ET_{č_1} = ET_{Z_1} + p\{T_{Z_1} < T_{X_2}\} \cdot ET_{X_2}$

čekanje na S_1

Z čeka da Y završi na S_1, a onda i da X završi na S_2, ako se i dalje opslužuje.

$ET_{č_2} = p\{T_{Z_1} < T_{Y_2}\} \cdot ET_{Y_2}$

Z i Y istovremeno kreću sa opsluživanjem na S_1, S_2 redom. Ako Z prvi završi, vreme se resetuje, i čeka da Y završi, kao da je krenuo ispočetka.

$$ET_u = \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_1} + \frac{\mu_1}{\mu_1+\mu_2} \cdot \frac{1}{\mu_2} + \frac{\mu_1}{\mu_1+\mu_2} \cdot \frac{1}{\mu_2}$$

$$= \frac{2}{\mu_1} + \frac{1}{\mu_2}\left(1 + \frac{2\mu_1}{\mu_1+\mu_2}\right)$$

II način:

$$ET_u = E(\max\{T_{Y_1}, T_{X_2}\}) + E(\max\{T_{Z_1}, T_{Y_2}\}) + ET_{Z_2}$$

$$= 2 \cdot \frac{1}{\mu_1+\mu_2}\left(1 + \frac{\mu_1}{\mu_2} + \frac{\mu_2}{\mu_1}\right) + \frac{1}{\mu_2}$$

11. a) $p\{X_1 < X_2 < X_3\} = p\{X_1 = \min\{X_1, X_2, X_3\}\} \cdot$
$p\{X_2 < X_3 \mid X_1 < X_2, X_1 < X_3\}$

$$= \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} \cdot p\{X_2 - X_1 < X_3 - X_1 \mid X_1 < X_2, X_1 < X_3\}$$

$$\overset{(JOM)}{=} \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} \cdot p\{X_2 < X_3\} = \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} \cdot \frac{\lambda_2}{\lambda_2+\lambda_3}$$

Neformalno, vreme se resetuje kad se X_1 završi, pa gledamo kao da su u tom trenutku X_2 i X_3 krenuli ispočetka.

b) $$p\{X_1 < X_2 \mid \max\{X_1, X_2, X_3\} = X_3\} = \frac{p\{X_1 < X_2 < X_3\}}{p\{X_1 < X_2 < X_3\} + p\{X_2 < X_1 < X_3\}}$$

$$= \frac{\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} \cdot \frac{\lambda_2}{\lambda_2+\lambda_3}}{\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} \cdot \frac{\lambda_2}{\lambda_2+\lambda_3} + \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3} \cdot \frac{\lambda_1}{\lambda_1+\lambda_3}} = \frac{\frac{1}{\lambda_2+\lambda_3}}{\frac{1}{\lambda_2+\lambda_3} + \frac{1}{\lambda_1+\lambda_3}}$$

v) $$E(\max\{X_1, X_2, X_3\} \mid X_1 < X_2 < X_3) = E(X_3 \mid X_1 < X_2 < X_3)$$

$$= E(X_1 + X_2 - X_1 + X_3 - X_2 \mid X_1 < X_2 < X_3)$$

$$= E(\min\{X_1, X_2, X_3\}) + E(X_2 + X_3 - X_2 \mid X_2 < X_3)$$

$$= \frac{1}{\lambda_1+\lambda_2+\lambda_3} + E(\min\{X_2, X_3\}) + EX_3$$

$$= \frac{1}{\lambda_1+\lambda_2+\lambda_3} + \frac{1}{\lambda_2+\lambda_3} + \frac{1}{\lambda_3}$$

Resetovali smo vreme kad se X_1 završio, pa ponovo kad se X_2 završio, tj. primenili smo JOM.