3 First-Step Analysis

3.1 Absorption Probability

Many functionals of homogeneous Markov chains, in particular probabilities of absorption by a closed set (A is called *closed* if $\sum_{j \in A} p_{ij} = 1$ for all $i \in A$) and average times before absorption, can be evaluated by a technique called *first-step analysis*. This technique, which is the motor of most computations in Markov chain theory, is best illustrated by the following example.

Example 3.1. *Gambler's Ruin*
Two players A and B play "heads or tails", where heads occur with probability $p \in (0, 1)$, and the successive outcomes form an i.i.d sequence. Calling X_n the fortune in dollars of player A at time n, then $X_{n+1} = X_n + Z_{n+1}$, where $Z_{n+1} = +1$ (resp., -1) with probability p (resp., $q = 1 - p$), and $\{Z_n\}_{n \geq 1}$ is i.i.d. In other words, A bets \$1 on heads at each toss, and B bets \$1 on tails. The respective initial fortunes of A and B are a and b. The game ends when a player is ruined, and therefore the process $\{X_n\}_{n \geq 1}$ is a random walk as described in Example 2.1, except that it is restricted to $E = \{0, \dots, a, a + 1, \dots, a + b = c\}$. The duration of the game is T, the first time n at which $X_n = 0$ or c, and the probability of winning for A is $u(a) = P(X_T = c \mid X_0 = a)$.

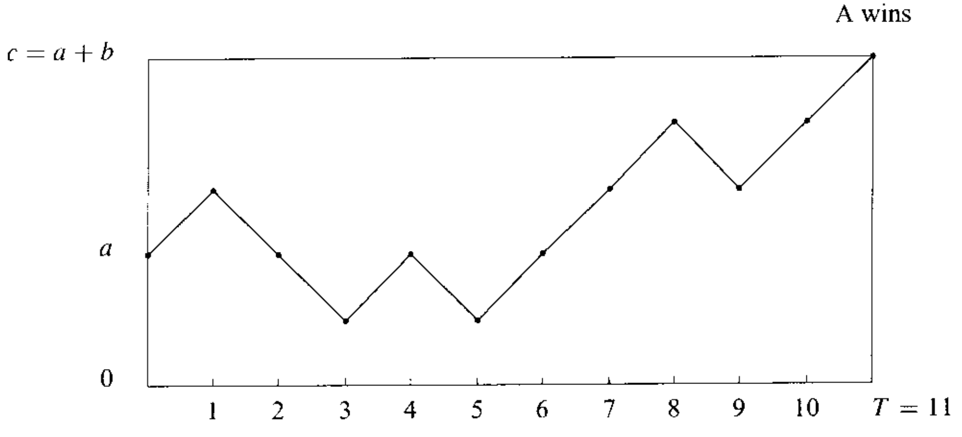

Figure 2.3.1. The basic random walk and the gambler's ruin

Instead of computing $u(a)$ alone, first-step analysis computes

$$u(i) = P(X_T = c \mid X_0 = i)$$

for all states $i \in [0, c]$, and for this, it first generates a recurrence equation for the $u(i)$'s by breaking down event "A wins" according to what can happen after the first step (the first toss) and using the rule of exclusive and exhaustive causes. If $X_0 = i \in [1, c - 1]$, then $X_1 = i + 1$ (resp., $X_1 = i - 1$) with probability p (resp., q), and the probability of ruin of

B starting with A's initial fortune $i+1$ (resp., $i-1$) is $u(i+1)$ (resp., $u(i-1)$). Therefore, for $i \in [1, c-1]$(see, however, a rigorous proof at the close of the example),

$$u(i) = pu(i+1) + qu(i-1), \tag{3.1}$$

with the boundary conditions

$$u(0) = 0, u(c) = 1.$$

The characteristic equation associated with this linear recurrence equation is $pr^2 - r + q = 0$. It has two distinct roots, $r_1 = 1$ and $r_2 = \frac{q}{p}$, if $p \neq q$, and a double root, $r_1 = 1$, if $p = q = \frac{1}{2}$. Therefore, the general solution is $u(i) = \lambda r_1^i + \mu r_2^i = \lambda + \mu \left(\frac{q}{p}\right)^i$ when $p \neq q$, and $u(i) = \lambda r_1^i + \mu i r_1^i = \lambda + \mu i$ when $p = q = \frac{1}{2}$. Taking into account the boundary conditions, one can determine the values of λ and μ. The result is, for $p \neq q$,

$$u(i) = \frac{1 - (\frac{q}{p})^i}{1 - (\frac{q}{p})^c}, \tag{3.2}$$

and for $p = q = \frac{1}{2}$,

$$u(i) = \frac{i}{c}. \tag{3.3}$$

In the case $p = q = \frac{1}{2}$, the probability $v(i)$ that B wins when the initial fortune of B is $c - i$ is obtained by replacing i by $c - i$ in expression (3.3): $v(i) = \frac{c-i}{c} = 1 - \frac{i}{c}$. One checks that $u(i) + v(i) = 1$, which means in particular that the probability that the game lasts forever is null. The reader is invited to check that the same is true in the case $p \neq q$. $\diamond$

A justification of the use of the rule of exclusive and exhaustive causes in the obtention of the recurrence equation (3.1) will be given, since we are not exactly in the standard framework of application. The same kind of proof can be performed for every instance of first-step analysis. However, needless to say, one should not go through it every time.

Let $Y = \{Y_n\}_{n \geq 0}$ denote the Markov chain obtained by delaying $X = \{X_n\}_{n \geq 0}$ by one time unit: $Y_n = X_{n+1}$. When $X_0 \in [1, c-1]$, the events "X is absorbed by 0" and "Y is absorbed by 0" are identical, and therefore

$$\begin{aligned} &P(X \text{ is absorbed by } 0,\ X_1 = i \pm 1,\ X_0 = i) \\ &\quad = P(Y \text{ is absorbed by } 0,\ X_1 = i \pm 1,\ X_0 = i). \end{aligned}$$

Since $\{Y_n\}_{n \geq 0}$ and X_0 are independent given X_1, the right-hand side of the above equality is equal to

$$P(X_0 = i, X_1 = i \pm 1) P(Y \text{ is absorbed by } 0 \mid Y_0 = i \pm 1).$$

The two chains have the same transition matrix, and therefore when they have the same initial state, they have the same distributions. Hence

$$P(Y \text{ is absorbed by } 0 \mid Y_0 = i \pm 1) = u(i \pm 1).$$

The rest of the proof is left for the reader.

Example 3.4. *Gambler's Ruin*
This example continues Example 3.1. The average duration $m(i) = E[T \mid X_0 = i]$ of the game when the initial fortune of player A is i satisfies the recurrence equation

$$m(i) = 1 + pm(i+1) + qm(i-1) \tag{3.4}$$

for $i \in [1, c-1]$. Indeed, the coin will be tossed at least once, and then with probability p (resp., q) the fortune of player A will be $i+1$ (resp., $i-1$), and therefore $m(i+1)$ (resp., $m(i-1)$) more tosses will be needed on average before one of the players goes broke. The boundary conditions are

$$m(0) = 0, m(c) = 0. \tag{3.5}$$

In order to solve (3.4) with the boundary conditions (3.5), write (3.4) in the form $-1 = p(m(i+1) - m(i)) - q(m(i) - m(i-1))$. Defining

$$y_i = m(i) - m(i-1),$$

we have, for $i \in [1, c-1]$,

$$-1 = py_{i+1} - qy_i \tag{3.6}$$

and

$$m(i) = y_1 + y_2 + \cdots + y_i. \tag{3.7}$$

We now solve (3.6) with $p = q = \frac{1}{2}$. From (3.6),

$$\begin{aligned} -1 &= \frac{1}{2}y_2 - \frac{1}{2}y_1, \\ -1 &= \frac{1}{2}y_3 - \frac{1}{2}y_2, \\ &\vdots \\ -1 &= \frac{1}{2}y_i - \frac{1}{2}y_{i-1}, \end{aligned}$$

and therefore, summing up,

$$-(i-1) = \frac{1}{2}y_i - \frac{1}{2}y_1,$$

that is, for $i \in [1, c]$,

$$y_i = y_1 - 2(i-1).$$

Reporting this expression in (3.7), and observing that $y_1 = m(1)$, we obtain

$$m(i) = im(1) - 2[1 + 2 + \cdots + (i-1)] = im(1) - i(i-1).$$

The boundary condition $m(c) = 0$ gives $cm(1) = c(c-1)$ and therefore, finally,

$$m(i) = i(c-i). \tag{3.8}$$

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