Varijacione metode za Puasonovu jednačinu

Zadatak 1

Metodom kolokacije rešiti granični problem:

$$-\Delta u = f(x,y), \quad (x,y) \in \Omega = (0,1) \times (0,1)$$

$$u = 0 \quad \text{na } \partial\Omega$$

gde je $f(x,y) = 2\pi^2 \sin(\pi x)\sin(\pi y)$.

Za bazne funkcije uzeti

$$\ell_1(x,y) = xy(1-x)(1-y)$$

$$\ell_2(x,y) = xy(1-x)(1-y)(2y-1)$$

$$\ell_3(x,y) = yx(1-x)(2x-1)(1-y)$$

$$\ell_4(x,y) = xy(1-x)(1-y)(2x-1)(2y-1)$$

a za tačke kolokacije:

$$\left(\tfrac{1}{3}, \tfrac{1}{3}\right), \quad \left(\tfrac{1}{3}, \tfrac{2}{3}\right), \quad \left(\tfrac{2}{3}, \tfrac{1}{3}\right), \quad \left(\tfrac{2}{3}, \tfrac{2}{3}\right)$$

Rešenje

Aproksimaciju tražimo u obliku:

$$v_4(x,y) = \sum_{i=1}^{4} c_i\, \ell_i(x,y)$$

i rešavamo sistem jednačina:

$$-\Delta v_4(x_i, y_i) = f(x_i, y_i), \quad i = \overline{1,4}$$

tj.

$$-\Delta\Big(\sum_{j=1} c_j \varphi_j(x_i, y_i)\Big) = f(x_i, y_i), \quad i = \overline{1,4}$$

Parcijalni izvodi baznih funkcija

$$\frac{\partial \ell_1}{\partial x} = y(1-y)(1-x) - xy(1-y) = (1-y)\big(y(1-x) - xy\big) = (1-y)(y-2xy)$$

$$\frac{\partial^2 \ell_1}{\partial x^2} = 2y(y-1), \qquad \frac{\partial^2 \ell_1}{\partial y^2} = 2x(x-1)$$

$$\frac{\partial^2 \ell_2}{\partial x^2} = 2y(y-1)(2y-1), \qquad \frac{\partial^2 \ell_2}{\partial y^2} = 6x(2y-1)(x-1)$$

$$\frac{\partial^2 \ell_3}{\partial x^2} = 6x(2x-1)(y-1), \qquad \frac{\partial^2 \ell_3}{\partial y^2} = 2x(2x-1)(x-1)$$

$$\frac{\partial^2 \ell_4}{\partial x^2} = 6y(2x-1)(2y^2-3y+1), \qquad \frac{\partial^2 \ell_4}{\partial y^2} = 6x(2y-1)(2x^2-3x+1)$$

Vrednosti u kolokacionim tačkama

Tačka $A = \left(\frac{1}{3}, \frac{1}{3}\right)$:

$$\ell_{1,xx}|_A = -\tfrac{4}{9}, \quad \ell_{1,yy}|_A = -\tfrac{4}{9}$$
$$\ell_{2,xx}|_A = \tfrac{4}{27}, \quad \ell_{2,yy} = \tfrac{4}{9}$$
$$\ell_{3,xx}|_A = \tfrac{4}{9}, \quad \ell_{3,yy} = \tfrac{4}{27}$$
$$\ell_{4,xx}|_A = -\tfrac{4}{27}, \quad \ell_{4,yy}|_A = -\tfrac{4}{27}$$

Tačka B=$\left(\frac{1}{3}, \frac{2}{3}\right)$:

$$\ell_{1,xx}|_B = -\tfrac{4}{9}, \quad \ell_{1,yy} = -\tfrac{4}{9}$$
$$\ell_{2,xx}|_B = -\tfrac{4}{27}, \quad \ell_{2,yy}|_B = -\tfrac{4}{9}$$
$$\ell_{3,xx}|_B = \tfrac{4}{9}, \quad \ell_{3,yy}|_B = \tfrac{4}{27}$$
$$\ell_{4,xx}|_B = \tfrac{4}{27}, \quad \ell_{4,yy}|_B = \tfrac{4}{27}$$

Tačka $C = \left(\frac{2}{3}, \frac{1}{3}\right)$:

$$\ell_{1,xx}|_C = -\tfrac{4}{9}, \quad \ell_{1,yy}|_C = -\tfrac{4}{9}$$
$$\ell_{2,xx}|_C = \tfrac{4}{27}, \quad \ell_{2,yy}|_C = \tfrac{4}{9}$$
$$\ell_{3,xx}|_C = -\tfrac{4}{9}, \quad \ell_{3,yy}|_C = -\tfrac{4}{27}$$
$$\ell_{4,xx}|_C = \tfrac{4}{27}, \quad \ell_{4,yy}|_C = \tfrac{4}{27}$$

Tačka $D = \left(\frac{2}{3}, \frac{2}{3}\right)$:

$$\ell_{1,xx}|_D = -\tfrac{4}{9}, \quad \ell_{1,yy} = -\tfrac{4}{9}$$
$$\ell_{2,xx}|_D = -\tfrac{4}{27}, \quad \ell_{2,yy}|_D = -\tfrac{4}{9}$$
$$\ell_{3,xx}|_D = -\tfrac{4}{9}, \quad \ell_{3,yy}|_D = -\tfrac{4}{27}$$
$$\ell_{4,xx}|_D = -\tfrac{4}{27}, \quad \ell_{4,yy}|_D = -\tfrac{4}{27}$$

Sistem jednačina (4×4)

Iz uslova $-\Delta v_4(x_i, y_i) = f(x_i, y_i) \iff -\Delta(\sum_{j=1}^{4} c_j \varphi_j(x_i, y_i)) = f(x_i, y_i), \quad i = 1, \dots 4,$ dobijamo sistem:

$$-\frac{8}{9}c_1 - \frac{16}{27}c_2 + -\frac{16}{27}c_3 + \frac{8}{27}c_4 = \pi^2 \cdot \frac{3}{4}$$
$$-\frac{8}{9}c_1 - \frac{16}{27}c_2 - \frac{16}{27}c_3 - \frac{8}{27}c_4 = \pi^2 \cdot \frac{3}{4}$$
$$\frac{8}{9}c_1 - \frac{16}{27}c_2 + \frac{16}{27}c_3 - \frac{8}{27}c_4 = \pi^2 \cdot \frac{3}{4}$$
$$\frac{8}{9}c_1 + \frac{16}{27}c_2 + \frac{16}{27}c_3 - \frac{8}{27}c_4 = \pi^2 \cdot \frac{3}{4}$$

Za desnu stranu sistema smo koristili:

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\sin\frac{2\pi}{3} = 2\sin\frac{\pi}{3}\cos\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Rešenje sistema

Iz sistema sledi:

$$c_2 = c_3 = c_4 = 0$$
$$c_1 = -\frac{\pi^2 \cdot 3 \cdot 9}{4 \cdot 8} = -\frac{\pi^2 \cdot 27}{32}$$

Dakle, aproksimacija rešenja je:

$$\boxed{u \approx v_4 = -\frac{27\pi^2}{32}\, xy(1-x)(1-y)}$$

Zadatak 2

Ricovom metodom rešiti granični problem:

$$-\Delta u = x(1+x)y(1-y), \quad (x, y) \in \Omega = (0,1)^2$$

$$u = 0 \quad \text{na } \partial\Omega$$

Za bazne funkcije uzeti:

$$\ell_1(x, y) = x(1-x)y(1-y)$$
$$\ell_2(x, y) = x(1-x)y(1-y)(2x-1)$$

Galerkinov sistem

$$\sum_{j=1}^{2} c_j \iint_\Omega \left(\frac{\partial \ell_j}{\partial x} \frac{\partial \ell_i}{\partial x} + \frac{\partial \ell_j}{\partial y} \frac{\partial \ell_i}{\partial y} \right) dx\, dy = \iint_\Omega f\, \ell_i\, dx\, dy, \quad i = \overline{1,2}$$

Matrica je simetrična. Sistem je 2×2:

$$A = [a_{ij}],\ i, j = \overline{1,2}, \qquad b = [b_i],\ i = \overline{1,2}$$

gde je:

$$a_{ij} = \iint_\Omega \frac{\partial \ell_j}{\partial x} \frac{\partial \ell_i}{\partial x} + \frac{\partial \ell_j}{\partial y} \frac{\partial \ell_i}{\partial y}\, dx\, dy, \qquad b_i = \iint_\Omega f\, \ell_i\, dx\, dy$$

$$\frac{\partial \ell_1}{\partial x} = (1-2x)y(1-y), \qquad \frac{\partial \ell_1}{\partial y} = (1-2y)x(1-x)$$

$$\frac{\partial \ell_2}{\partial x} = y(1-y)\big(-6x^2 + 6x - 1\big)$$

$$\frac{\partial \ell_2}{\partial y} = x(1-x)(2x-1)(1-2y) = x(2y-1)(2x^2 - 3x + 1)$$

Rešenje aproksimiramo kao:

$$u \approx v_2 = c_1\, \ell_1(x,y) + c_2\, \ell_2(x,y)$$

gde se koeficijenti c_1, c_2 dobijaju rešavanjem sistema $Ac = b$. Rešavanjem sistema dobijamo da je $c_1 = \frac{1}{20}$ i $c_2 = 0$. pa je $v_2 = \frac{1}{20} x(1-x)y(1-y)$.