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(1) $f(z) = f(x+iy) = u(x,y) + iv(x,y)$

KP: $u_x = -v_y$ и $u_y = v_x$ $\boxed{1}$

$$|f(x,y)| = \sqrt{u^2+v^2} \quad \boxed{1}$$

смер $\Leftarrow$ лако јер $\Delta(\text{const}) = 0$ $\boxed{1}$

смер $\Rightarrow$:

Ако $\exists (x,y)$ тд. $|f(x,y)| = 0$, онда $|f|$ је харм. и достиже мин $\Rightarrow$ const $\boxed{2}$

Даље, $\sqrt{u^2+v^2} > 0$, па су изводи деф.

$$|f|_{xx} = \left(\frac{1}{\cancel{2}\sqrt{u^2+v^2}} \cdot (\cancel{2}uu_x + \cancel{2}vv_x)\right)_x =$$

$$= \frac{1}{\sqrt{u^2+v^2}}\left(u_x^2 + uu_{xx} + v_x^2 + vv_{xx}\right) +$$

$$(uu_x + vv_x) \cdot \left(-\frac{1}{\cancel{2}}\right) \cdot (u^2+v^2)^{-3/2} \cdot (\cancel{2}uu_x + \cancel{2}vv_x)$$

$$\Delta|f| = \frac{u_x^2 + u_y^2 + v_x^2 + v_y^2 + u\overset{=0}{\Delta u} + v\overset{=0}{\Delta v}}{|f|} - \frac{(uu_x + vv_x)^2 + (uu_y + vv_y)^2}{|f|^3} =$$

$$= \frac{u_x^2 + u_y^2 + v_x^2 + v_y^2}{|f|} - \frac{u^2(u_x^2+u_y^2) + v^2(v_x^2+v_y^2) + 2uv\overset{=0}{(u_xv_x + u_yv_y)}}{|f|^3}$$

$$= \frac{u_x^2 + u_y^2 + v_x^2 + v_y^2}{|f|} - \frac{u_x^2 + v_x^2}{|f|} = \frac{u_y^2 + v_y^2}{|f|} = 0 \quad \boxed{7}$$

$\Rightarrow u_y = v_y = u_x = v_x = 0 \Rightarrow f = \text{const} \in \mathbb{C}$ $\boxed{1}$

(2) $\Delta u(x,y) = \dfrac{2}{(1+y)^3}$

$u_p(x,y) = \dfrac{1}{1+y} \to$ ограничена на Ω

$$\Delta u_p = \left(\frac{1}{1+y}\right)_{yy} = \left(-\frac{1}{(1+y)^2}\right) = \frac{2}{(1+y)^3}$$

$v = u - u_p$ смена $\qquad \Delta v = 0$ на Ω

$$v|_{\partial\Omega} = u|_{\partial\Omega} - u_p|_{\partial\Omega} = \frac{y^2+y+1-1}{y+1} = y \quad \boxed{3}$$

$w(\rho,\varphi) = v(\rho\cos\varphi, \rho\sin\varphi) = R(\rho)\cdot\Phi(\varphi)$, $\rho \geq 1$, $\varphi \in [0,\pi]$

$\Delta v = 0 \Rightarrow \rho^2 w_{\rho\rho} + \rho w_\rho + w_{\varphi\varphi} = 0$

$\rho^2 R''\Phi + \rho R'\Phi + R\Phi'' = 0$

$$\frac{\rho^2R'' + \rho R'}{R} = -\frac{\Phi''}{\Phi} = \lambda$$

$\Phi'' + \lambda\Phi = 0$, $\Phi(0) = \Phi(\pi) = 0$ $\boxed{1}$

$\lambda > 0$: $\Phi(\varphi) = c_1\sin(\sqrt{\lambda}\varphi) + c_2\cos(\sqrt{\lambda}\varphi)$

$\varphi = 0$. $c_2\cos(\sqrt{\lambda}\cdot 0) = 0 \Rightarrow c_2 = 0$

$\varphi = \pi$: $c_1\sin(\sqrt{\lambda}\pi) = 0 \Rightarrow \sqrt{\lambda}\pi = k\pi \Rightarrow \lambda = k^2$

$\Phi_k(\varphi) = c_1\cdot\sin(k\varphi)$, $k \in \mathbb{N}$ $\boxed{2}$

$\rho^2R_k'' + \rho R_k' - k^2R_k = 0 \Rightarrow R_k(\rho) = a_k\rho^k + b_k\rho^{-k}$

$$w(\rho,\varphi) = \sum_{k=1}^{\infty}(a_k\rho^k + b_k\rho^{-k})\sin(k\varphi)$$

$$a_k = 0 \Rightarrow w(\rho,\varphi) = \sum_{k=1}^{\infty} b_k\rho^{-k}\sin(k\varphi) \quad \boxed{3}$$

$\rho = 1$, $\varphi \in [0,\pi]$: $w|_{\rho=1} = y = 1\cdot\sin\varphi$

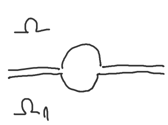
$b_1 = 1$, $b_k = 0$, $k \geq 2$ $\boxed{2}$

$$w(\rho,\varphi) = 1\cdot\rho^{-1}\cdot\sin\varphi = \frac{\sin\varphi}{\rho} = \frac{\rho\sin\varphi}{\rho^2}$$

$$v(x,y) = \frac{y}{x^2+y^2} \Rightarrow u(x,y) = \frac{1}{1+y} + \frac{y}{x^2+y^2} \quad \boxed{1}$$

$$v \leq \frac{1}{1+0} + \frac{y}{\sqrt{x^2+y^2}} \leq 2 \Rightarrow \text{ограничена је} \quad \boxed{1}$$

јединствености: $\Delta v = 0$, $v|_{\partial\Omega} = y$



сведено на проблем на спољашњ. диска:

$$v_1(x,y) = \begin{cases} v(x,y), & (x,y) \in \Omega \\ -v(x,-y), & (x,y) \in \Omega_1 \end{cases}$$

v_1 решава: $\Delta v_1 = 0$ на $\{x^2+y^2 > 1\}$

$v_1 = y$ на $\{x^2+y^2 = 1\}$ $\boxed{5}$

v_1 - ограничена $\rightsquigarrow$ јединст.

(3) $u = \underset{\text{хомог.}}{\alpha} + \underset{\text{нехом.}}{\beta}$ $\boxed{1}$

$$\alpha(x,t) = \sum_{k=1}^{\infty}(a_k\cos 2kt + b_k\sin 2kt)\cdot\sin 2kx$$

$\alpha = 0$ из поч. услова $\boxed{1}$

$\beta_{tt} - \beta_{xx} = \cos 2t \cdot x(\pi - 2x)\pi$

$$\beta(x,t) = \sum_{k=1}^{\infty} T_k(t)\cdot\sin 2kx \quad \boxed{2}$$

$$\sum_{k=1}^{\infty}(T_k'' + 4k^2T_k)\sin 2kx = \sum_{k=1}\cos 2t\cdot A_k\cdot\sin 2kx \quad \boxed{1}$$

$$A_k = \frac{2}{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} x(\pi-2x)\pi\cdot\sin 2kx\,dx =$$

$$= 4\left(\pi\cdot\frac{\pi(-1)^{k+1}}{4k} - \frac{(2-\pi^2k^2)(-1)^k - 2}{4k^3}\right) =$$

$$= 4\left(-\frac{\pi^2(-1)^k}{4k} - \frac{(-1)^k}{2k^3} + \frac{\pi^2(-1)^k}{4k} + \frac{1}{2k^3}\right) =$$

$$= \frac{2}{k^3}\left(1 - (-1)^k\right) \quad \boxed{4}$$

$$T_k'' + 4k^2T_k = \cos 2t\cdot\frac{2}{k^3}\left(1-(-1)^k\right)$$

$T_k(0) = T_k'(0) = 0$ $\boxed{1}$

$k = 1$: $T_1(t) = c_1\cos 2t + c_2\sin 2t + t(c_3\sin 2t + c_4\cos 2t)$

$\vdots$

$T_1(t) = t\sin 2t$ $\boxed{2}$

$k \neq 1$: $T_k(t) = c_1\cos 2kt + c_2\sin 2kt + (c_3\sin 2t + c_4\cos 2t)$

$$T_k(t) = \frac{\cos 2t - \cos 2kt}{k^3(k^2-1)} \quad \boxed{2}$$

Решење:

$$u(x,t) = t\sin 2t\cdot\sin 2x + \sum_{k=2}^{\infty}\frac{\cos 2t - \cos 2kt}{k^3(k^2-1)}\sin 2kx$$

још: класе C^2 и задовољава услове $\boxed{4}$

јединствености: теорема $\boxed{1}$