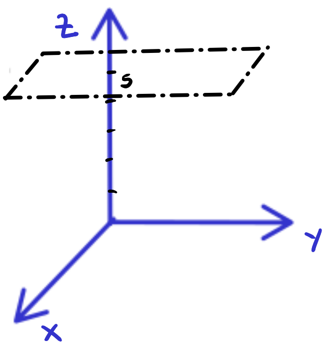
4.8. Одредити растојање тачака A и B преко њихових поларних координата $A(r_A, \phi_A)$, $B(r_B, \phi_B)$.

$$A(\overbrace{r_A\cos\phi_A}^{=x_A},\ \overbrace{r_A\sin\phi_A}^{=y_A}),\quad B(\overbrace{r_B\cos\phi_B}^{=x_B},\ \overbrace{r_B\sin\phi_B}^{=y_B})$$

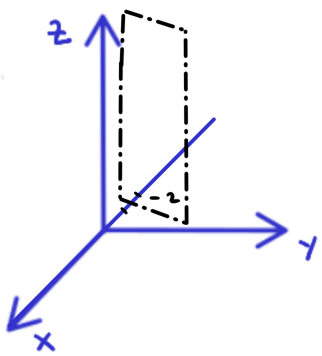
$$d(A,B) = \sqrt{(x_A-x_B)^2+(y_A-y_B)^2}$$
$$= \sqrt{(r_A\cos\phi_A - r_B\cos\phi_B)^2 + (r_A\sin\phi_A - r_B\sin\phi_B)^2}$$
$$= \sqrt{r_A^2 + r_B^2 - 2r_Ar_B(\cos\phi_A\cos\phi_B + \sin\phi_A\sin\phi_B)}$$
$$= \boxed{\sqrt{r_A^2+r_B^2-2r_Ar_B\cos(\phi_A-\phi_B)}}$$

4.9. Скицирати раван а) $z-5=0$; б) $x=-2$; в) $y+z-3=0$; г) $x+2y+3z-6=0$.

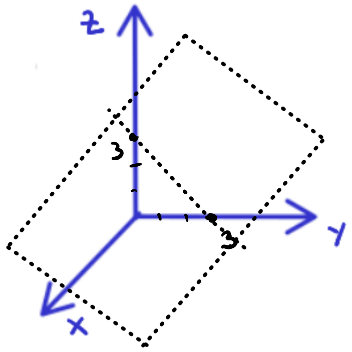
а) $z-5=0$



б) $x=-2$

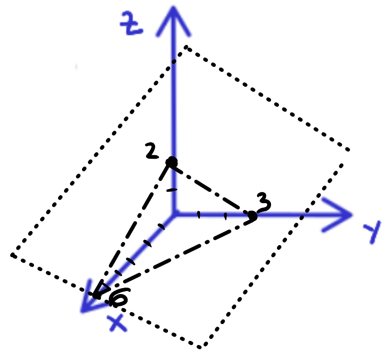


в) $y+z-3=0$



г) $x+2y+3z=6$

$\frac{x}{6}+\frac{y}{3}+\frac{z}{2} = 1$

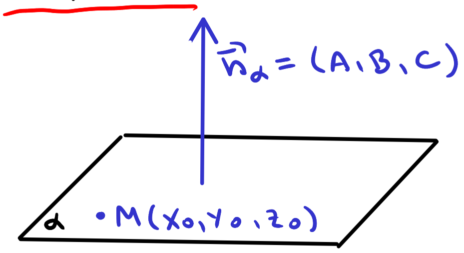


4 Тачка, права и раван

* У ПРОСТОРУ $\mathbb{R}^3$

* РАВАН

$$\alpha:\ Ax+By+Cz+D=0 \Rightarrow \vec{n}_\alpha = (A,B,C)$$

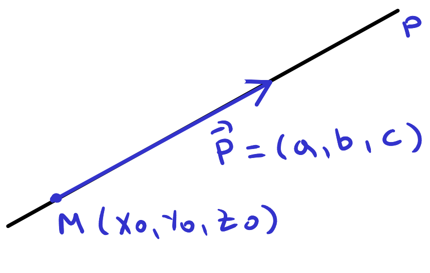


$$\Rightarrow \alpha:\ A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$

* ПРАВА

M ∈ p

$$\Rightarrow p:\ \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} = t,\qquad (a,b,c) = \vec{p}$$



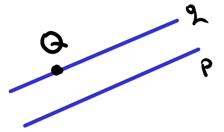
$$p:\ \begin{cases} x = at + x_0 \\ y = bt + y_0 \\ z = ct + z_0 \end{cases},\quad t\in\mathbb{R}$$

$\vec{p}$ M

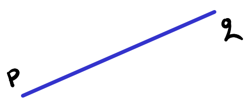
$$p:\ \begin{cases} A_1x+B_1y+C_1z+D_1=0 \\ A_2x+B_2y+C_2z+D_2=0 \end{cases}$$

* ОДНОС ДВЕ ПРАВЕ

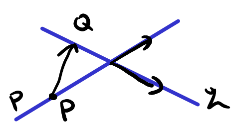
$p \Rightarrow \vec{p},\ P\in p,\quad q \Rightarrow \vec{q},\ Q\in q$



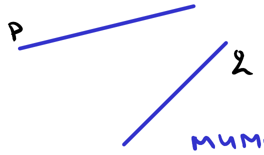
паралелне p∥q



поклапају се



секу се



мимоилазне

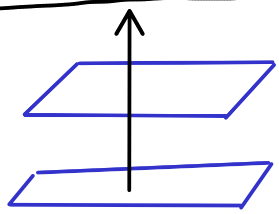
$$\vec{p} \times \vec{q} = \vec{0} \qquad\qquad \vec{p} \times \vec{q} \neq \vec{0}$$

$Q \notin p$ $\qquad$ $Q \in p$ $\qquad$ $[\vec{p}, \vec{q}, \vec{PQ}] = 0$ $\qquad$ $[\vec{p}, \vec{q}, \vec{PQ}] \neq 0$

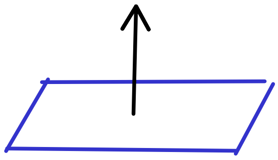
$\Rightarrow$

$$\boxed{\begin{array}{ll} 1)\ p \parallel q \Rightarrow \vec{p} = \vec{q} & 2)\ p \perp q \Rightarrow \vec{p} \perp \vec{q} \\ 3)\ p \text{ и } q \text{ секу} \Rightarrow [\vec{p}, \vec{q}, \vec{PQ}] = 0 & \end{array}}$$

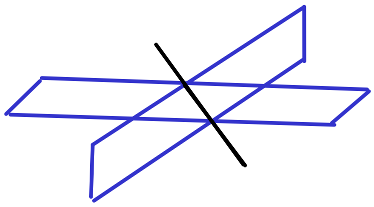
* **ОДНОС ДВЕ РАВНИ** $\qquad$ $\alpha \Rightarrow \vec{n}_\alpha,\ A \in \alpha,\quad \beta \Rightarrow \vec{n}_\beta,\ B \in \beta$



$\alpha \parallel \beta$



$\alpha \equiv \beta$



α и β се секу

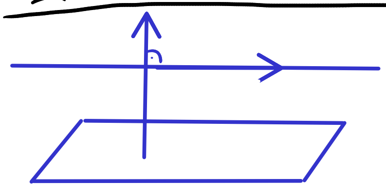
$\vec{n}_\alpha \times \vec{n}_\beta \neq \vec{0}$

$$\vec{n}_\alpha \times \vec{n}_\beta = \vec{0}$$

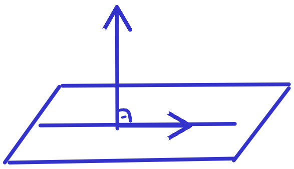
$A \notin \beta$ $\qquad$ $A \in \beta$

$$\boxed{4)\ \alpha \parallel \beta \Rightarrow \vec{n}_\alpha = \vec{n}_\beta \qquad 5)\ \alpha \perp \beta \Rightarrow \vec{n}_\alpha \perp \vec{n}_\beta}$$

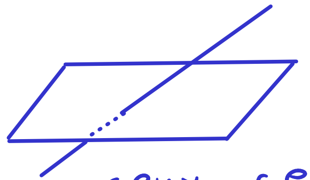
* **ОДНОС ПРАВЕ И РАВНИ** $\qquad$ $\alpha \Rightarrow \vec{n}_\alpha,\ A \in \alpha,\quad p \Rightarrow \vec{p},\ P \in p$



$p \parallel \alpha$



$p \subset \alpha$



секу се

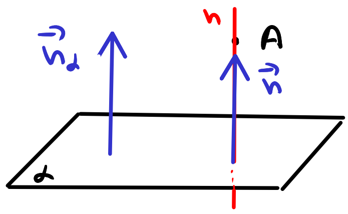
$\vec{n}_\alpha \circ \vec{p} \neq 0$

$$\vec{n}_\alpha \circ \vec{p} = 0$$

$P \notin \alpha$ $\qquad$ $P \in \alpha$

$$\boxed{6)\ \begin{array}{l} p \parallel \alpha \\ p \subset \alpha \end{array} \Rightarrow \vec{p} \perp \vec{n}_\alpha \qquad 7)\ p \perp \alpha \Rightarrow \vec{p} = \vec{n}_\alpha}$$

4.10. Одредити једначину нормале из тачке $A(2, 3, -1)$ на раван $\alpha : 2x + y - 4z + 5 = 0$.

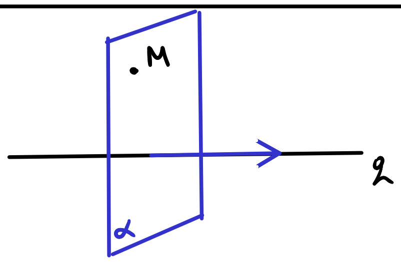


$A(2,3,-1) \in n$ ✓

$n \perp \alpha \Rightarrow \vec{n} = \vec{n}_\alpha = (2,1,-4)$ ✓

$$\Rightarrow \boxed{n : \frac{x-2}{2} = \frac{y-3}{1} = \frac{z+1}{-4}}$$

4.11. Одредити једначину равни која садржи тачку $M(-1, 0, 3)$ и нормална је на праву $q : \frac{x+1}{2} = \frac{y-3}{4} = \frac{z-3}{-1}$.



$M(-1,0,3) \in \alpha$ ✓

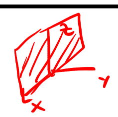
$\alpha \perp q \Rightarrow \vec{n}_\alpha = \vec{q} = (2,4,-1)$ ✓

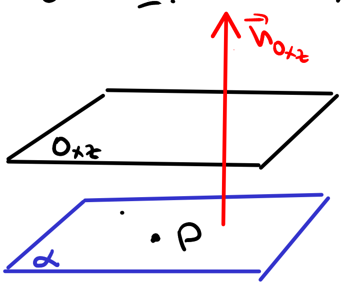
$\Rightarrow \alpha : 2(x+1) + 4(y-0) - 1(z-3) = 0$

$$\boxed{\alpha : 2x + 4y - z + 5 = 0}$$

4.12. Одредити једначину равни која:
а) је паралелна са равни Oxz и садржи тачку $P(2,3,5)$ б) садржи z-осу и тачку $M(-3,1,2)$ в) паралелна је x оси и садржи тачке $A(4,0,-2)$, $B(5,1,7)$.

а) α т.д. $\alpha \parallel O_{xz}$, $P(2,3,5) \in \alpha$?





$P \in \alpha$ ✓

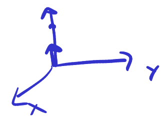
$O_{xz} : y = 0$

$\alpha \parallel O_{xz} \Rightarrow \vec{n}_\alpha = \vec{n}_{O_{xz}} = (0,1,0)$ ✓

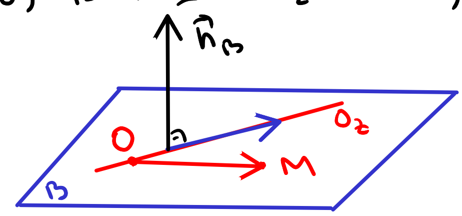
$\Rightarrow \alpha : 0(x-2) + 1(y-3) + 0(z-5) = 0$

$$\boxed{\alpha : y = 3}$$

б) β т.д. $O_z \subset \beta$, $M(-3,1,2) \in \beta$?



$O(0,0,0) \in O_z$



$M \in \beta$ ✓

$O_z : x=0, y=0, z=t, t \in \mathbb{R}$, $O_z : \frac{x}{0} = \frac{y}{0} = \frac{z}{1}$

1) $O_z \subset \beta \Rightarrow \vec{n}_\beta \perp \vec{O}_z = (0,0,1)$

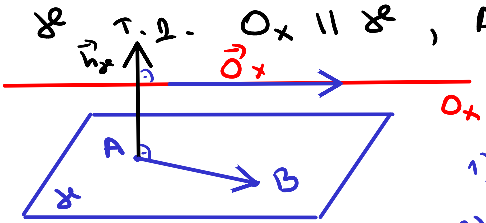
2) $\vec{OM} \subset \beta \Rightarrow \vec{n}_\beta \perp \vec{OM} = (-3,1,2)$

$$\vec{n}_\beta = \vec{OM} \times \vec{O}_z = \begin{vmatrix} i & j & k \\ -3 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = (1,3,0) \ ✓$$

$\Rightarrow \beta : 1(x+3) + 3(y-1) + 0(z-2) = 0$

$$\boxed{\beta : x + 3y = 0}$$

в) γ т.д. $O_x \parallel \gamma$, $A(4,0,-2) \in \gamma$, $B(5,1,7) \in \gamma$?



$A \in \gamma$ ✓ $O_x : y=0, z=0, x=t, t \in \mathbb{R}$

1) $\gamma \parallel O_x \Rightarrow \vec{n}_\gamma \perp \vec{O}_x = (1,0,0)$

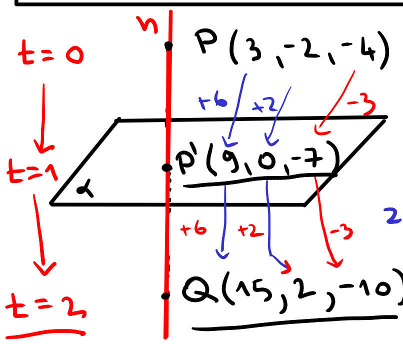
2) $\vec{AB} \subset \gamma \Rightarrow \vec{n}_\gamma \perp \vec{AB} = (1,1,9)$

$$\Rightarrow \vec{n}_\gamma = \vec{AB} \times \vec{O}_x = \begin{vmatrix} i & j & k \\ 1 & 1 & 9 \\ 1 & 0 & 0 \end{vmatrix} = (0,9,-1) \ ✓$$

$\Rightarrow \gamma : 0(x-4) + 9(y-0) - 1(z+2) = 0$

$$\boxed{\gamma : 9y - z - 2 = 0}$$

4.13. Одредити тачку Q која је симетрична тачки $P(3,-2,-4)$ у односу на раван $\alpha : 6x+2y-3z-75=0$ као и пројекцију P' тачке P на раван α.



1) n т.д. $P \in n$, $n \perp \alpha \Rightarrow \vec{n} = \vec{n}_\alpha = (6,2,-3)$

$\Rightarrow n : \frac{x-3}{6} = \frac{y+2}{2} = \frac{z+4}{-3} = t \Rightarrow n : \begin{cases} x = 6t+3 \\ y = 2t-2 \\ z = -3t-4 \end{cases}, t \in \mathbb{R}$

2) $P' = n \cap \alpha$:

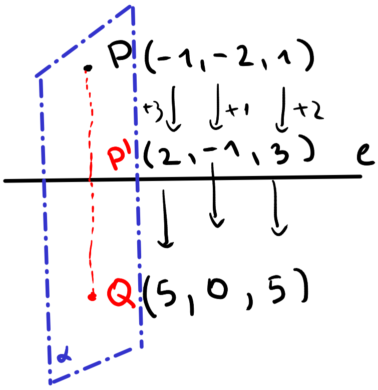
$6(6t+3) + 2(2t-2) - 3(-3t-4) - 75 = 0$

$49t - 49 = 0 \Rightarrow t = 1 \Rightarrow \boxed{P'(9,0,-7)}$

3) $P' = S(PQ) \Rightarrow [P'] = \frac{[P]+[Q]}{2}$

$[Q] = 2[P'] - [P] \Rightarrow \boxed{Q(15,2,-10)}$

4.14. Одредити тачку Q која је симетрична тачки $P(-1,-2,1)$ у односу на праву $l : \frac{x}{-2} = \frac{y-3}{4} = \frac{z-4}{1}$ $=t$ као и пројекцију P' тачке P на праву l.



1) α т.д. $P \in \alpha$, $\alpha \perp l \Rightarrow \vec{n}_\alpha = \vec{l} = (-2,4,1)$

$\alpha : -2(x+1)+4(y+2)+1(z-1) = 0$

$$\boxed{\alpha : -2x+4y+z+5=0}$$

2) $P' = \alpha \cap l :$ $\quad l: \begin{cases} x = -2t \\ y = 4t+3 \\ z = t+4 \end{cases}, \ t \in \mathbb{R}$

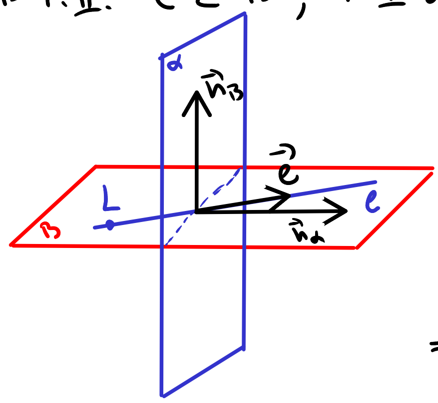
$4t+16t+12+t+4+5=0 \Rightarrow 21t+21=0$

$\Rightarrow \boxed{t=-1} \Rightarrow \boxed{P'(2,-1,3)}$

3) $\boxed{Q(5,0,5)}$

4.15. Одредити једначину равни која садржи праву $l : \frac{x-1}{2} = \frac{y+2}{1} = \frac{z-3}{3}$ и нормална је на раван $\alpha : 2x-4y+z+5 = 0$.

β т.д. $l \subset \beta$, $\beta \perp \alpha$



1) $L(1,-2,3) \in l \subset \beta$ ✓

2) $l \subset \beta \Rightarrow \vec{n}_\beta \perp \vec{l} = (2,1,3)$

$\beta \perp \alpha \Rightarrow \vec{n}_\beta \perp \vec{n}_\alpha = (2,-4,1)$

$\Rightarrow \vec{n}_\beta = \vec{n}_\alpha \times \vec{l} = \begin{vmatrix} i & j & k \\ 2 & -4 & 1 \\ 2 & 1 & 3 \end{vmatrix} = (-13,-4,10)$ ✓

$\Rightarrow \beta : -13(x-1) - 4(y+2) + 10(z-3) = 0$

$$\boxed{\beta : -13x - 4y + 10z - 25 = 0}$$

4.16. Одредити λ тако да се праве $p : \frac{x-2}{3} = \frac{y+4}{5} = \frac{z-1}{-2}$ $=t$ и $q : \frac{x-\lambda}{2} = \frac{y-3}{1} = \frac{z+5}{0}$ $=s$! секу. Које су координате пресечне тачке?

$\boxed{\text{I}}$ p и q секу $\Rightarrow [\vec{p}, \vec{q}, \overrightarrow{PQ}] = 0 \Rightarrow \dots \ \lambda = \dots$

$\boxed{\text{II}}$ $p: \begin{cases} x = 3t+2 \\ y = 5t-4 \\ z = -2t+1 \end{cases}, \ t \in \mathbb{R}$ $\qquad q: \begin{cases} x = 2s+\lambda \\ y = s+3 \\ z = -5 \end{cases}, \ s \in \mathbb{R}$

$p \cap q :$ $\begin{cases} 3t+2 = 2s+\lambda \\ 5t-4 = s+3 \\ -2t+1 = -5 \end{cases}$ $\Rightarrow \underline{t=3}$

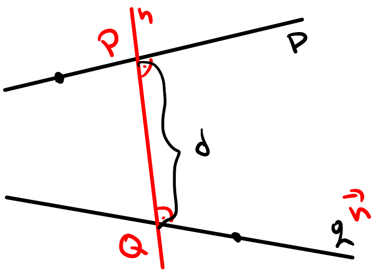
$\to 9+2 = 16+\lambda \Rightarrow \boxed{\lambda = -5}$

$15-4 = s+3 \Rightarrow \underline{s = 8}$

$$\boxed{p \cap q = A(11,11,-5)}$$

4.17. Одредити заједничку нормалу и растојање између мимоилазних правих $p : \frac{x-4}{1} = \frac{y+3}{2} = \frac{z-12}{-1}$ $=t$ и $q : \frac{x-3}{-7} = \frac{y-1}{2} = \frac{z-1}{3}$ $=s$

$\vec{p} = (1,2,-1)$ $\qquad \vec{q} = (-7,2,3)$



$P(t+4, 2t-3, -t+12) \in p$ произв.

$Q(-7s+3, 2s+1, 3s+1) \in q$ произв.

$\vec{n} = \overrightarrow{PQ} = (-7s-t-1, 2s-2t+4, 3s+t-11)$

1) $\vec{PQ} \circ \vec{p} = 0 \Rightarrow -7s - t - 1 + 4s - 4t + 8 - 3s - t + 11 = 0$

$\Rightarrow \boxed{-6s - 6t = -18}$ ①

2) $\vec{PQ} \circ \vec{q} = 0 \Rightarrow 49s + 7t + 7 + 4s - 4t + 8 + 9s + 3t - 33 = 0$

$\boxed{62s + 6t = 18}$ ②

① + ② $\Rightarrow 56s = 0 \Rightarrow \boxed{s = 0} \Rightarrow \boxed{t = 3}$

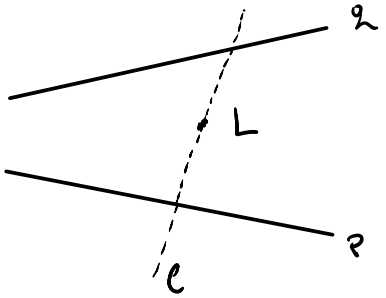
$\Rightarrow P(7,3,9)\,,\ Q(3,1,1)\,,\ \vec{PQ} = (-4,-2,-8) \parallel (2,1,4)$

1) $\boxed{n = PQ : \dfrac{x-7}{2} = \dfrac{y-3}{1} = \dfrac{z-9}{4}}$!

2) $d(p,q) = \|\vec{PQ}\| \overset{!}{=} \sqrt{(-4)^2 + (-2)^2 + (-8)^2} = \sqrt{84} = \boxed{2\sqrt{21}}$

$\boxed{\text{II}}$ $d(p,q) = \dfrac{|[\vec{p}, \vec{q}, \vec{PQ}]|}{\|\vec{p} \times \vec{q}\|} = \cdots = 2\sqrt{21}$

4.18. Одредити једначину праве која садржи тачку $L(2,-1,7)$ и сече праве $p : \frac{x-1}{2} = \frac{y-4}{-3} = \frac{z-3}{1}$ и $q : \frac{x-7}{-1} = \frac{y-11}{-3} = \frac{z+2}{0}$.



$L(2,-1,7) \in e$ ✓, $\vec{e} = (a,b,c)$

$P(1,4,3) \in p\,,\ \vec{p} = (2,-3,1)$

$Q(7,11,-2)\,,\ \vec{q} = (-1,-3,0)$

1) e и q се секу $\Rightarrow [\vec{e}, \vec{q}, \vec{LQ}] = 0$

$\Rightarrow 0 = \begin{vmatrix} a & b & c \\ -1 & -3 & 0 \\ 5 & 12 & -9 \end{vmatrix} = 27a - 9b + 3c \Rightarrow 9a - 3b + c = 0$

2) e и p се секу $\Rightarrow [\vec{e}, \vec{p}, \vec{LP}] = 0$

$\Rightarrow 0 = \begin{vmatrix} a & b & c \\ 2 & -3 & 1 \\ -1 & 5 & -4 \end{vmatrix} = 7a + 7b + 7c \Rightarrow a + b + c = 0 \quad / \cdot (-1)$ +

$$\begin{aligned} a + b + c &= 0 \\ 8a - 4b &= 0 \end{aligned} \Rightarrow \begin{aligned} c &= -3a \\ b &= 2a \end{aligned} \Rightarrow \vec{e} = (a, 2a, -3a)$$

$a = 1 \Rightarrow \vec{e} = (1,2,-3)$ ✓

$\Rightarrow \boxed{e : \dfrac{x-2}{1} = \dfrac{y+1}{2} = \dfrac{z-7}{-3}}$