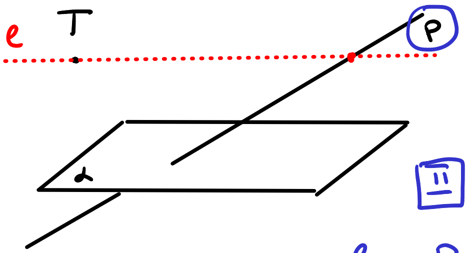
4.19. Кроз тачку $T(-3,1,2)$ одредити праву l која је паралелна равни $\alpha: 4x-y+2z-5=0$ и која сече праву $p: \frac{x+3}{0}=\frac{y-2}{2}=\frac{z+1}{-1}$. $= t$



$\boxed{\text{I}}$ НАЧИН 1) $T(-3,1,2)\in l$ ✓ $\quad \vec{l}=(a,b,c)$

2) $l \parallel \alpha \Rightarrow \vec{l}\circ\vec{n}_\alpha = 0$

3) l и p се секу $\Rightarrow [\vec{l},\vec{p},\vec{PT}]=0$, $P(-3,2,-1)$

$\boxed{\text{II}}$ НАЧИН $\quad P(-3,2t+2,-t-1)\in p$ произв.

$l = PT \Rightarrow T\in l$, l сече p ✓

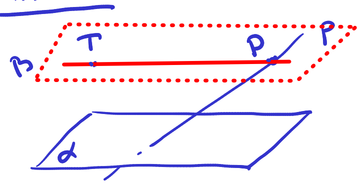
$l\parallel\alpha \Rightarrow \vec{n}_\alpha\circ\vec{l}=0$, $\vec{n}_\alpha=(4,-1,2)$, $\vec{l}=\vec{PT}=(0,-2t-1,t+3)$

$\Downarrow$ $2t+1+2t+6=0 \Rightarrow t=\frac{-7}{4}$

$\vec{l}=\left(0,\frac{5}{2},\frac{5}{4}\right)\parallel(0,2,1)$

$$\Rightarrow \boxed{l: \frac{x+3}{0}=\frac{y-1}{2}=\frac{z-2}{1}}$$

$\boxed{\text{III}}$ НАЧИН



1) β т.д. $T\in\beta$, $\beta\perp\alpha$

2) $\beta\cap p = P$

3) $l = PT$

4.20. Одредити раван α која са равни $\gamma: x-4y-8z+12=0$ образује угао $\frac{\pi}{4}$ и садржи праву: а) $x+5y+z=0, x-z+4=0$ б) $p: \frac{x-1}{1}=\frac{y-2}{0}=\frac{z}{-1}$.

а) α т.д. $\measuredangle(\alpha,\gamma)=\frac{\pi}{4}$, $\alpha\supset p: \begin{cases} x+5y+z=0 \\ x-z+4=0\end{cases}$

(Т) Прамен равни које садрже праву $p=\alpha_1\cap\alpha_2$ је облика $\mathcal{X}_{\alpha_1,\alpha_2}: \lambda_1\alpha_1+\lambda_2\alpha_2=0$.

$$\Rightarrow \boxed{\alpha: \lambda_1(x+5y+z)+\lambda_2(x-z+4)=0} \Rightarrow p\subset\alpha$$

2) $\measuredangle(\gamma,\alpha)=\frac{\pi}{4}$, $\vec{n}_\gamma=(1,-4,-8)$

2.1) $\lambda_1=0 \Rightarrow \alpha: x-z+4=0$? $\quad \vec{n}_\alpha=(1,0,-1)$

$$\cos\measuredangle(\alpha,\gamma)=\frac{|\vec{n}_\alpha\circ\vec{n}_\gamma|}{\|\vec{n}_\alpha\|\cdot\|\vec{n}_\gamma\|}=\frac{|1+8|}{\sqrt{2}\cdot\sqrt{81}}=\frac{9}{\sqrt{2}\cdot 9}=\frac{\sqrt{2}}{2}$$

$$\Rightarrow \measuredangle(\alpha,\gamma)=\frac{\pi}{4} \Rightarrow \boxed{\alpha_1: x-z+4=0}$$

2.2) $\lambda_1\neq 0 \Rightarrow \alpha: x+5y+z+\lambda(x-z+4)=0$

$$\alpha: (1+\lambda)x+5y+(1-\lambda)z+4\lambda=0$$

$$\frac{\sqrt{2}}{2}=\cos\measuredangle(\alpha,\gamma)=\frac{|1+\lambda-20-8+8\lambda|}{9\sqrt{(1+\lambda)^2+25+(1-\lambda)^2}}=\frac{|9\lambda-27|}{9\sqrt{2\lambda^2+27}}=\frac{|\lambda-3|}{\sqrt{2\lambda^2+27}}$$

$$\sqrt{2}\sqrt{2\lambda^2+27}=2|\lambda-3| \quad /^2$$

$$\cancel{4\lambda^2} + 54 = 4(\lambda^2 - 6\lambda + 9) = \cancel{4\lambda^2} - 24\lambda + 36$$

$$24\lambda = -18 \Rightarrow \boxed{\lambda = -\frac{3}{4}}$$

$$\Rightarrow \alpha_2 : \frac{1}{4}x + 5y + \frac{7}{4}z - 3 = 0 \quad / \cdot 4$$

$$\boxed{\alpha_2 : x + 20y + 7z - 12 = 0}$$

б) $p: \frac{x-1}{1} = \frac{y-2}{0} = \frac{z}{-1}$.

$$\frac{x-1}{1} = \frac{y-2}{0} \Rightarrow y - 2 = 0$$

$$\frac{x-1}{1} = \frac{z}{-1} \Rightarrow z = -x + 1$$

α т.д. $\measuredangle(\alpha, \gamma) = \frac{\pi}{4}$, $p \subset \alpha$

I начин

$$\boxed{p: \begin{array}{l} y - 2 = 0 \\ x + z - 1 = 0 \end{array}} \Rightarrow \alpha : \lambda_1(y-2) + \lambda_2(x+z-1) = 0$$

...

II начин

1) $P(1,2,0) \in p \subset \alpha$ ✓, $\vec{n}_\alpha = (A, B, C)$

2) $p \subset \alpha \Rightarrow \vec{n}_\alpha \circ \vec{p} = 0 \Rightarrow (A,B,C) \circ (1,0,-1) = 0$

$$\Rightarrow A - C = 0 \Rightarrow \boxed{C = A} \Rightarrow \vec{n}_\alpha = (A, B, A)$$

3) $\measuredangle(\alpha, \gamma) = \frac{\pi}{4}$, $\vec{n}_\gamma = (1, -4, -8)$

$$\Rightarrow \frac{\sqrt{2}}{2} = \cos\measuredangle(\alpha,\gamma) = \frac{|\vec{n}_\alpha \circ \vec{n}_\gamma|}{\|\vec{n}_\alpha\| \cdot \|\vec{n}_\gamma\|} = \frac{|A - 4B - 8A|}{9\sqrt{A^2 + B^2 + A^2}}$$

$$9\sqrt{2}\sqrt{2A^2 + B^2} = 2|-7A - 4B| \quad /^2$$

$$81 \cdot \cancel{2} \cdot (2A^2 + B^2) = \cancel{4}^{2}(49A^2 + 56AB + 16B^2)$$

$$162A^2 + 81B^2 = 98A^2 + 112AB + 32B^2$$

$$64A^2 - 112AB + 49B^2 = 0$$

$/ : B^2 \neq 0 \Rightarrow$ ~~квадратна~~ ...

$$(8A - 7B)^2 = 0$$

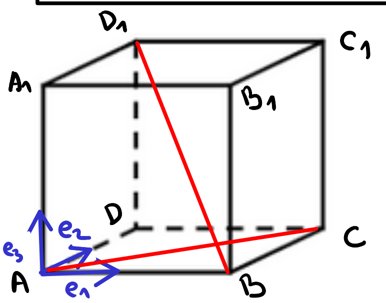
$$\Rightarrow 8A = 7B \Rightarrow \boxed{B = \frac{8}{7}A}$$

$$\Rightarrow \vec{n}_\alpha = \left(A, \frac{8}{7}A, A\right) \parallel (7, 8, 7) \quad ✓$$

$$\alpha : 7(x-1) + 8(y-2) + 7(z-0) = 0$$

$$\boxed{\alpha : 7x + 8y + 7z - 23 = 0}$$

3.7. Дата је коцка $ABCDA_1B_1C_1D_1$ ивице a. Одредити а) угао између мимоилазних правих AC и BD_1; б) растојање између правих AC и BD_1; в) угао између праве AC и равни $\alpha = BCD_1$.



$$\boxed{Ae : \vec{e}_1 = \frac{\vec{AB}}{a}, \ \vec{e}_2 = \frac{\vec{AD}}{a}, \ \vec{e}_3 = \frac{\vec{AA_1}}{a}}$$

$\Rightarrow A(0,0,0)$, $C(a,a,0)$, $B(a,0,0)$, $D_1(0,a,a)$

а) $$\cos\measuredangle(AC, BD_1) = \frac{|\vec{AC} \circ \vec{BD_1}|}{\|\vec{AC}\| \cdot \|\vec{BD_1}\|} = \frac{|(a,a,0) \circ (-a,a,a)|}{\sqrt{2a^2} \cdot \sqrt{3a^2}} = 0$$

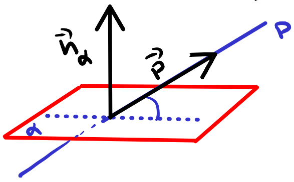
$$\Rightarrow \boxed{AC \perp BD_1}$$

б) $d(AC, BD_1) = \dfrac{|[\vec{AC}, \vec{BD_1}, \vec{AB}]|}{\|\vec{AC} \times \vec{BD_1}\|} = \dfrac{\left|\begin{vmatrix} a & a & 0 \\ -a & a & a \\ a & 0 & 0 \end{vmatrix}\right|}{\left\|\begin{vmatrix} e_1 & e_2 & e_3 \\ a & a & 0 \\ -a & a & a \end{vmatrix}\right\|} = \dfrac{|a^3|}{\|(a^2, -a^2, 2a^2)\|}$

$\begin{vmatrix} a & a & 0 \\ -a & a & a \\ a & 0 & 0 \end{vmatrix} = a \cdot (-1)^{2+3} \cdot \begin{vmatrix} a & a \\ a & 0 \end{vmatrix} = -a \cdot (-a^2) = a^3$

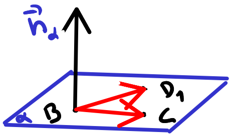
$$d(AC, BD_1) = \frac{a^3}{\sqrt{6a^4}} = \frac{a^3}{\sqrt{6}\, a^2} = \frac{a}{\sqrt{6}} = \boxed{\frac{a\sqrt{6}}{6}}$$

в) $p = AC$, $\alpha = BCD_1$



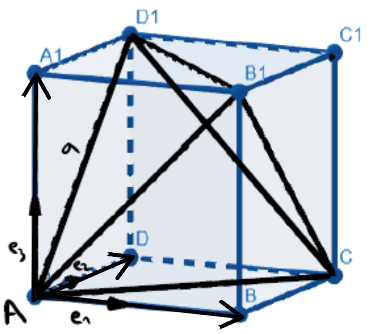
$$\sin \measuredangle(p, \alpha) = \frac{|\vec{n}_\alpha \circ \vec{p}|}{\|\vec{n}_\alpha\| \cdot \|\vec{p}\|} = \frac{a^3}{\sqrt{2a^4} \cdot \sqrt{2a^2}} = \frac{a^3}{2a^3} = \frac{1}{2}$$

$\vec{p} = \vec{AC} = (a, a, 0)$, $\vec{n}_\alpha = \vec{BC} \times \vec{BD_1} = \begin{vmatrix} e_1 & e_2 & e_3 \\ 0 & a & 0 \\ -a & a & a \end{vmatrix} = (a^2, 0, a^2)$



$\Rightarrow \boxed{\measuredangle(AC, BCD_1) = \frac{\pi}{6}}$

4.21. Одредити угао између пљосни правилног тетраедра.



$Ae: \ \vec{e_1} = \vec{AB}, \ \vec{e_2} = \vec{AD}, \ \vec{e_3} = \vec{AA_1}$

тетраедар уписан у коцку ивице 1

$A(0,0,0)$, $C(1,1,0)$, $B_1(1,0,1)$, $D_1(0,1,1)$

$\alpha = ACB_1$, $\beta = AB_1D_1$

$\vec{n}_\alpha = \vec{AC} \times \vec{AB_1} = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = (1, -1, -1)$

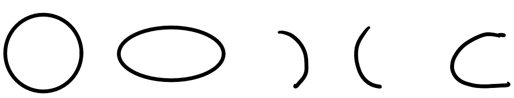
$\vec{n}_\beta = \vec{AB_1} \times \vec{AD_1} = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = (-1, -1, 1)$

$\cos \measuredangle(\alpha, \beta) = \dfrac{|\vec{n}_\alpha \circ \vec{n}_\beta|}{\|\vec{n}_\alpha\| \cdot \|\vec{n}_\beta\|} = \dfrac{|-1|}{\sqrt{3} \cdot \sqrt{3}} = \dfrac{1}{3} \Rightarrow \boxed{\measuredangle(\alpha, \beta) = \arccos \frac{1}{3}}$

5 Криве другог реда

$\mathcal{K}: \ a_{11} x^2 + 2a_{12} xy + a_{22} y^2 + 2a_{13} x + 2a_{23} y + a_{33} = 0$

* недегенерисане



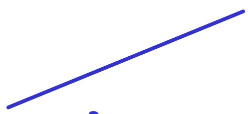
* дегенерисане

$\emptyset$

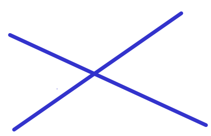
$x^2 + y^2 + 1 = 0$

$\cdot$

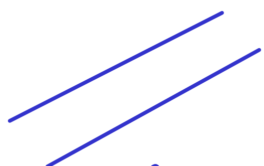
$x^2 + y^2 = 0$



$y^2 = 0$



$(x + y + 1)(x - y) = 0$

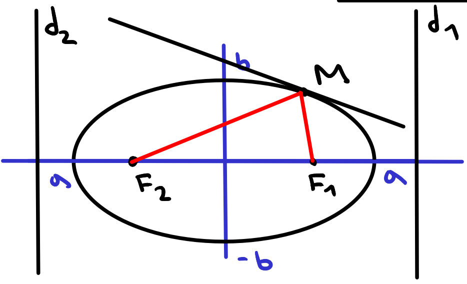


$(x - y)^2 - 4 = 0$

. . .

Елипса

$$\mathcal{E}: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$, a, b - полуосе , $a > b$



$c = \sqrt{a^2 - b^2}$, $e = \frac{c}{a}$, $\boxed{0 < e < 1}$

жиже $F_1(c, 0)$, $F_2(-c, 0)$

директрисе $d_{1/2}: x = \pm \frac{a}{e} \iff x = \pm \frac{a^2}{e}$

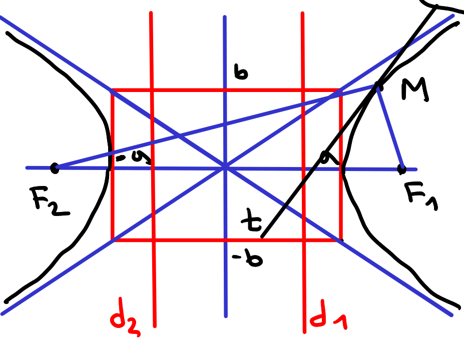
$\mathcal{E}: \frac{d(M,F)}{d(M,d)} = e$, $MF_1 + MF_2 = 2a$, $M \in \mathcal{E}$

* $M(x_0, y_0) \in \mathcal{E} \Rightarrow$ $t_M: \frac{x_0}{a^2} x + \frac{y_0}{b^2} y = 1$

* Спец. за $a = b \Rightarrow$ круг $(e = 0)$

Хипербола

$$\mathcal{X}: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$, a, b - полуосе



$c = \sqrt{a^2 + b^2}$, $e = \frac{c}{a}$, $\boxed{e > 1}$

жиже $F_1(c, 0)$, $F_2(-c, 0)$

директрисе $d_{1/2}: x = \pm \frac{a}{e}$

$\frac{d(M,F)}{d(M,d)} = e$, $|MF_1 - MF_2| = 2a$, $M \in \mathcal{X}$

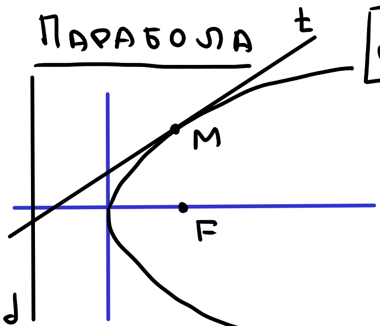
асимптоте $a_{1/2}: y = \pm \frac{b}{a} x \iff \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = 0$

* $M(x_0, y_0) \in \mathcal{X} \Rightarrow$ $t_M: \frac{x_0}{a^2} x - \frac{y_0}{b^2} y = 1$

Парабола

$$\mathcal{P}: y^2 = 2px$$, $p > 0$, p - параметар $\mathcal{P}$



$\boxed{e = 1}$

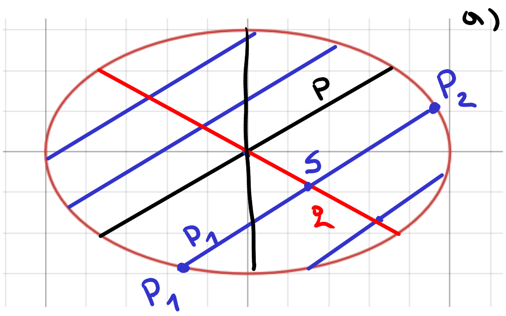
жижа $F\left(\frac{p}{2}, 0\right)$

директриса $d: x = -\frac{p}{2}$

$d(M, F) = d(M, d)$, $M \in \mathcal{P}$

* $M(x_0, y_0) \in \mathcal{P} \Rightarrow$ $t_M: y_0 \cdot y = p(x + x_0)$

5.1. а) Доказати да средишта тетива паралелних дијаметру p елипсе (хиперболе) припадају некој правој q која пролази кроз центар елипсе (хиперболе). б) Доказати да су тангенте у крајевима дијаметра p паралелне правој q. в) Доказати да су (у случају елипсе) тангете у крајевима дијаметра q паралелне дијаметру p.



а) $\mathcal{K}: \frac{x^2}{a^2} + \mu \frac{y^2}{b^2} = 1$ $/a^2b^2$

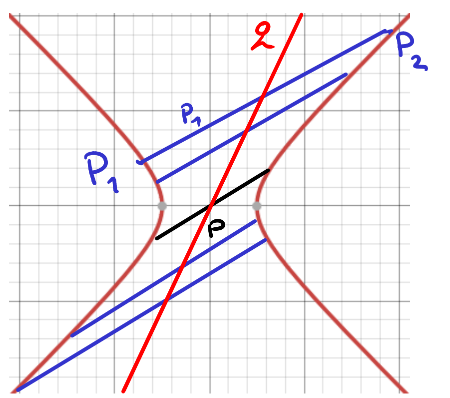
$\mu = 1 \Rightarrow \mathcal{E}$, $\mu = -1 \Rightarrow \mathcal{X}$

1) $p: x = 0 \Rightarrow$ очигледно

2) $p: y = k_p x$

$p_1 \parallel p \Rightarrow p_1: y = k_p x + n$

$P_1(x_1, y_1), P_2(x_2, y_2) \in p_1 \cap \mathcal{K}$



$P_1 \cap \mathcal{K}: \quad b^2x^2 + \mu a^2 y^2 = a^2 b^2$

$y = k_p x + n$

$$b^2x^2 + \mu a^2\left(k_p^2x^2 + 2k_p x n + n^2\right) = a^2b^2$$

$$\underbrace{\boxed{\left(b^2 + \mu a^2 k_p^2\right)}}_{A} x^2 + \underbrace{\boxed{2\mu a^2 k_p n}}_{B}\, x + \underbrace{\boxed{\mu a^2 n^2 - a^2 b^2}}_{C} = 0$$

/ $ax^2 + bx + c = 0 \overset{\text{Вијет}}{\Longrightarrow} x_1 + x_2 = -\frac{b}{a}, \quad x_1 \cdot x_2 = \frac{c}{a}$ (*) /

$$S(P_1P_2) = S\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$x_s = \frac{x_1+x_2}{2} \overset{(*)}{=} -\frac{B}{2A} = -\frac{2\mu a^2 k_p n}{2\left(b^2+\mu a^2 k_p^2\right)} = -\frac{\mu a^2 k_p n}{b^2 + \mu a^2 k_p^2}$$

$$y_s = k_p x_s + n = \frac{\cancel{-k_p \mu a^2 k_p n} + n b^2 + \cancel{\mu n a^2 k_p^2}}{b^2 + \mu a^2 k_p^2} = \frac{n b^2}{b^2 + \mu a^2 k_p^2}$$

$$y_s = \frac{n b^2}{b^2 + \mu a^2 k_p^2} \cdot \frac{-\mu a^2 k_p}{-\mu a^2 k_p} = \frac{b^2}{-\mu a^2 k_p} \cdot \overbrace{\boxed{\frac{-\mu a^2 k_p n}{b^2 + \mu a^2 k_p^2}}}^{\text{“} x_s} = -\frac{b^2}{\mu a^2 k_p}\, x_s$$

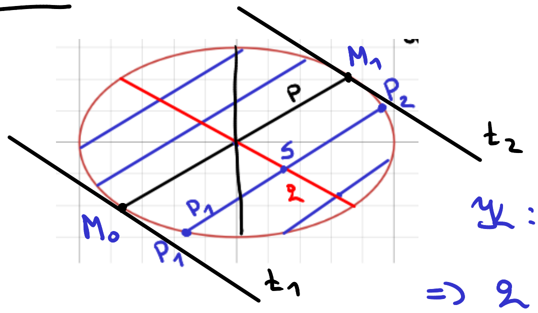
$\Rightarrow$ сва средишта припадају правој $\boxed{q: y = \frac{-b^2}{\mu a^2 k_p}\, x}$

Деф Дијаметри $p: y = k_p x$ и $q: y = k_q x$ за које важи $\boxed{k_p \cdot k_q = -\frac{b^2}{\mu a^2}}$ називају се узајамно конјуговани дијаметри.

$$\boxed{\mathcal{E} \Rightarrow k_p k_q = -\frac{b^2}{a^2}, \qquad \mathcal{H} \Rightarrow k_p k_q = \frac{b^2}{a^2}}$$

Спец. O_x и O_y-осе су узајамно конј. дијаметри.

б)

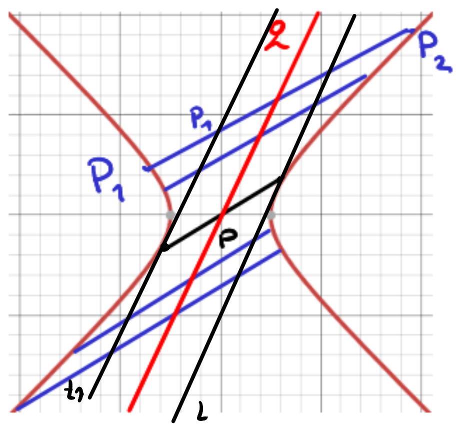


$\boxed{p: y = k_p x}$

$\mathcal{K}: \frac{x^2}{a^2} + \mu \frac{y^2}{b^2} = 1$

$\mathcal{K}: b^2x^2 + \mu a^2 y^2 = a^2 b^2$

$\Rightarrow q: y = \underbrace{\left(-\frac{b^2}{\mu a^2 k_p}\right)}_{= k_q} x$



$p \cap \mathcal{K} = M(x_0, y_0)$

1) $M \in \mathcal{K} \Rightarrow t_M: b^2 \cdot x_0 x + \mu a^2 y_0 y = a^2 b^2$

$$t: y = -\frac{b^2 x_0}{\mu a^2 y_0}\, x + \frac{a^2 b^2}{\mu a^2 y_0} \Rightarrow k_t = -\frac{b^2 x_0}{\mu a^2 y_0}$$

2) $M \in p \Rightarrow y_0 = k_p x_0 \Rightarrow \frac{x_0}{y_0} = \frac{1}{k_p}$

$$\Rightarrow k_t = -\frac{b^2}{\mu a^2 k_p} = k_q \Rightarrow \boxed{t_M \parallel q}$$

в) аналогно уз замену k_p и k_q

Задатак $\boxed{5.2}$ за вежбу (аналогно зад. 5.1)